Two Closely Coupled Lasers
dynamics & applications

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Abstract

The dynamics of two mutually coupled identical single-mode semi-conductor lasers are theoretically investigated. For small separation and large coupling between the lasers, symmetry-broken one-colour states are shown to be stable. In this case the light output of the lasers have significantly different intensities whilst at the same time the lasers are locked to a single common frequency. For intermediate coupling we observe stable two-colour states, where both single-mode lasers lase simultaneously at two optical frequencies which are separated by up to 150 GHz. For low coupling but possibly large separation, the frequency of the relaxation oscillations of the free-running lasers defines the dynamics. Chaotic and quasi-periodic states are identified and shown to be stable. For weak coupling undamped relaxation oscillations dominate where each laser is locked to three or more odd number of colours spaced by the relaxation oscillation frequency. It is shown that the instabilities that lead to these states are directly connected to the two colour mechanism where the change in the number of optical colours due to a change in the plane of oscillation.

At initial coupling, in-phase and anti-phase one colour states are shown to emerge from “on” uncoupled lasers using a perturbation method. Similarly symmetry-broken one-colour states come from considering one free-running laser initially “on” and the other laser initially “off”. The mechanism that leads to a bi-stability between in-phase and anti-phase one-colour states is understood. Due to an equivariant phase space symmetry of being able to exchange the identical lasers, a symmetric and symmetry-broken variant of all states mentioned above exists and is shown to be stable. Using a five dimensional model we identify the bifurcation structure which is responsible for the appearance of symmetric and symmetry-broken one-colour, symmetric and symmetry-broken two-colour, symmetric and symmetry-broken undamped relaxation oscillations, symmetric and symmetry-broken quasi-periodic, and symmetric and symmetry-broken chaotic states. As symmetry-broken states always exist in pairs, they naturally give rise to bi-stability. Several of these states show multi-stabilities between symmetric and symmetry-broken variants and among states.

Three memory elements on the basis of bi-stabilities in one and two colour states for two coupled single-mode lasers are proposed. The switching performance of selected designs of optical memory elements is studied numerically.
Contents

Abstract i

Declaration ii

Dedication iii

Contents iv

Preface vi

Acknowledgements vii

Abbreviations viii

Symbols ix

Reference x

Appendices

2.A Perturbation method ............................................. 19
3.A Symmetric and Symmetry Broken .................................. 38
3.B The frequency $\omega_A$ for one-colour states ....................... 39
3.C The frequencies $\omega_A$ and $\omega_B$ .................................. 39
3.D Anti-Phase Linearization ........................................ 40
4.A Pitchfork-Hopf codimension two points .............................. 55
4.B Small non-zero delay ............................................. 56
1 Introduction .......................... 1
  1.1 High Coupling .................................. 2
  1.2 Low Coupling .................................. 3
  1.3 Applications .................................. 4

2 The Model ................................ 6
  2.1 Delay-Differential Description .................. 8
  2.2 Initial Coupling ................................ 9
  2.3 Reduced Coordinate System .................... 12

3 Dynamics Observed ......................... 23
  3.1 One-Colour States .............................. 23
  3.2 Two-Colour States .............................. 24
  3.3 Undamped Relaxation Oscillations ............... 25
  3.4 Quasi-periodic ................................ 27

4 Bifurcation Picture ......................... 43
  4.1 High Coupling ................................ 43
  4.2 Low Coupling ................................ 45

5 All-Optical Switching ......................... 59
  5.1 Optical Memory Unit 1 ........................ 59
  5.2 Optical Memory Unit 2 ........................ 60
  5.3 Optical Memory Unit 3 ........................ 61

6 Conclusions ................................ 67
  6.1 High Coupling ................................ 67
  6.2 Low Coupling ................................ 67
  6.3 Memory Element ............................... 68

7 Outlook ................................ 69
  7.1 Medium Coupling ............................... 69
  7.2 Random Number Generator ..................... 70
Preface

Let's begin by considering the classical origins of the words dissertation and thesis. The word “dissertation” comes from the Latin dissertare meaning to examine or to discuss. The etymology of the word “thesis” comes from the ancient Greek equivalent to the Latin-derived word “proposition”. Reports presented for a PhD degree are generally described using one of these names. Although flexibility in definition is required due to the scope of modern research work, traditionally at least, it’s still supposed to represent a single idea which is to be addressed in full and communicated logically. For the most part, the main chapters of this thesis tie together two dynamical systems studies for mutually coupled single-mode lasers, one at low coupling between the lasers and one at high coupling. These studies tell a complimentary story which fit together nicely, not as two separate chunks of works which they were but as a single body giving a coherent dissertation about a single topic. I’ve tried not to mess with this by adding additional research material accomplished over the last 4 years which may have increased the page count but would bring a reader away from the central concept. In short, this thesis or dissertation gives the most complete to date theoretical description of the dynamics of two identical mutually coupled semiconductor lasers in face-to-face configuration with a delay or coupling strength small enough so that complications due to the delay do not occur.

The dissertation consists of 7 chapters, 50 figures, 120 references, 90 mathematical expressions with a total of 30,000 words. It can of course be read linearly from start to finish. Each chapter is laid out as follows

(a) Main Text     (b) Main Figures     (c) Appendix     (d) Appendix Figures

with some minor exceptions of some schematic diagrams embedded in the text. Most readers may not be interested in delving too deep into the material and, as such, the appendices may be skipped without detracting significantly from the central technical results of the thesis. There are also three principal bodies of work in this dissertation which can be understood relatively independent of the other sections. For quick reads we suggest the following routes

(i) high coupling     Secs. 1.1 → 2.1 → 2.3 → 3.1 → 3.2 → 4.1 → 6.1
(ii) low coupling     Secs. 1.2 → 2.1 → 2.2 → 3.3 → 3.4 → 4.2 → 6.2
(iii) applications     Secs. 1.3 → 5.0 → 5.1 → 5.2 → 5.3 → 6.3 → 7.2

For readers intending to get a more complete dynamical systems understanding, the medium coupling Sec. 7.1 should be included as it goes some way to tie together the high (i) and low (ii) coupling studies. The proposed applications would not have been possible without the knowledge developed by studying the dynamical regimes (i) and (ii), however the sections were written in such a way to help people skip as much as possible. Our proposal of three all-optical memory elements based upon coupled semiconductor lasers are supplemented by Sec. 7.2 which presents a candidate design for an optoelectronic true random number generator.
Abbreviations

App.            appendix
Ar+            Argon ion
C_{32}H_{18}AlCIN_{8}  Chloroaluminum phthalocyanine
Chap.          chapter
cf.            confer
CLM(s)         Compound Laser Mode(s)
CO_2           Carbon dioxide
CPU            Central Processing Unit
DBR            distributed Bragg reflector
ECM(s)         External Cavity Mode(s)
Eq(s)          equation(s)
Fig(s)         figure(s)
GaAs          Gallium Arsenide
GH            Generalised Hopf (Bautin) bifurcation
H             Hopf bifurcation
H_{sb}        Hopf bifurcation acting on symmetry-broken states
HeNe         Helium-Neon
i.e.          id est
LHS           left-hand side
NH_3          Ammonia
ODE(s)        ordinary differential equation(s)
P            pitchfork bifurcation
Pat.          patent
PD            period-doubling bifurcation
PDT          period-doubling torus co-dimension two bifurcation
PH[1/2]      pitchfork-Hopf [1/2] co-dimension two bifurcation
PIC          Photonic Integrated Circuit
PLC          pitchfork of limit cycles bifurcation
PRNG          Pseudo Random Number Generator
QW            quantum well
Ref(s).       reference(s)
RHS          right-hand side
RNG          Random Number Generator
Sec(s).       section(s)
SNH          saddle-node Hopf bifurcation
SNP          saddle-node pitchfork (Bautin) bifurcation
TRNG         True Random Number Generator
VCSEL       vertical-cavity surface-emitting laser
ZH            zero Hopf bifurcation
Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>line-width enhancement factor</td>
</tr>
<tr>
<td>$\delta$</td>
<td>phase difference between lasers</td>
</tr>
<tr>
<td>$\delta_A$</td>
<td>phase difference between laser for one colour states</td>
</tr>
<tr>
<td>$\bar{\delta}$</td>
<td>second phase difference for two colour states</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>frequency detuning between lasers</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>small positive number</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>coupling strength</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>free-running optical wavelength</td>
</tr>
<tr>
<td>$\Lambda_{E,P,N}$</td>
<td>damping of the fields ($E$, $P$ and $N$)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>delay-time between the lasers</td>
</tr>
<tr>
<td>$\tau_i$</td>
<td>minimal period of limit cycle</td>
</tr>
<tr>
<td>$\tau_p$</td>
<td>photon lifetime</td>
</tr>
<tr>
<td>$\phi_{1,2}$</td>
<td>absolute phases in laser 1,2</td>
</tr>
<tr>
<td>$\phi_{A,D,S}$</td>
<td>(Average</td>
</tr>
<tr>
<td>$\omega$</td>
<td>angular frequency</td>
</tr>
<tr>
<td>$\omega_\Delta$</td>
<td>free-running frequency</td>
</tr>
<tr>
<td>$\omega_A,B,C,D$</td>
<td>first, second, third, fourth detuned optical frequency</td>
</tr>
<tr>
<td>$\omega_H$</td>
<td>characteristic angular frequency at Hopf bifurcation</td>
</tr>
<tr>
<td>$\omega_{in,an,ab}$</td>
<td>(in-phase</td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>rapidly oscillating physical field</td>
</tr>
<tr>
<td>$\Omega_{E,P}$</td>
<td>(slowly varying) oscillation in complex field ($E</td>
</tr>
<tr>
<td>$A_{1,2}$</td>
<td>amplitude for a one-colour state</td>
</tr>
<tr>
<td>$A_{S,D}$</td>
<td>(sum</td>
</tr>
<tr>
<td>$B_{1,2}$</td>
<td>second amplitude for a two-colour state</td>
</tr>
<tr>
<td>$c$</td>
<td>speed of light</td>
</tr>
<tr>
<td>$C_p$</td>
<td>coupling phase</td>
</tr>
<tr>
<td>$D$</td>
<td>distance between lasers</td>
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<td>$E_{1,2}$</td>
<td>electric field in laser 1,2</td>
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<td>inversions in laser 1,2</td>
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<tr>
<td>$N_{D,S}$</td>
<td>(Difference</td>
</tr>
<tr>
<td>$P$</td>
<td>pumping</td>
</tr>
<tr>
<td>$P$</td>
<td>slow varying complex polarisation field</td>
</tr>
<tr>
<td>$q_{x,y,z}$</td>
<td>q-vector space</td>
</tr>
<tr>
<td>$S^1$</td>
<td>symmetric group 1</td>
</tr>
<tr>
<td>$T$</td>
<td>ratio carrier/photon lifetimes</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>reflection symmetry</td>
</tr>
<tr>
<td>$Z_n$</td>
<td>cyclic group n</td>
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Bibliography


Free Software Foundation. Gnu operating system, 2014.


xvi


1 Introduction

Light Amplification by Stimulated Emission of Radiation was experimentally realised using a ruby medium by Theodore H. Maiman on the 16th of May 1960 at Hughes Research laboratories [72]. Several research groups [101, 53, 36] were working with the hope that stimulated emission of photons would yield a spectrally pure, amplified, highly directional light and their efforts were rewarded with lasers based on HeNe [54], Ar+ [16], CO2 [91], GaAs [2], C32H14AlClN4 [105] media successful demonstrated within a handful of years. A big break through for the semiconductor laser [9], which is the focus of this thesis, came with introduction of a heterostructure to confine electrons and holes in an optical wave-guide [24]. We direct the interested reader to the fascinating history of the invention of the laser recalled in Refs. [45, 3]. Early in this story, it was recognised that small amounts of disruption whether from an external source or even feedback of the laser’s light from a nearby surface could destroy the coherence of the light, a prized property of lasers. In the interest of avoiding any deviation in output, an understanding of the rich dynamics, simply referred to as “coherence collapse” [66] or “instabilities” [96], was motivated. A dynamical system’s study was warranted. The subject of “laser dynamics” was born.

To understand the effect of stray light incident on the laser facet from a miscellaneous source, a setup schematised in Fig. 1 is studied [122]. In this scenario, a second, normally tuneable, laser mimics the external source in a controlled way. Rich dynamical phenomena is reported including: frequency locking, chirp, bursting, frequency oscillations, and chaos [59]. The feedback scenario, where the laser’s light is reabsorbed after some delay, is a different problem. An experimental and theoretical scenario schematised in Fig. 2 was explored for many decades [65]. The basic single frequency constant amplitude solutions are known as external cavity modes (ECMs) in the literature [44]. Examples of more complex dynamics include: undamped relaxation oscillations, mode beating, and chaos [116]. The scenario studied in this thesis of two mutually coupled identical lasers (Fig. 3) is a natural progression in complexity and could naïvely be considered a simple coupling back of the second laser in the injection problem or the replacing of the mirror with a second identical laser in the feedback problem. It is true that the mutually coupled lasers inherits many aspects from both. It is shown in this thesis that while similarities exist, the extra degree of freedom that comes from having a second laser influences the dynamics considerably. For example, the single frequency constant-amplitude states are called constant-phase compound laser modes (CLMs) in Ref. [27]. These are directly analogous to ECMs of the feedback problem. However as two laser exists, both lasers can have the same inversions and optical intensity but be completely “in-phase” or completely “anti-phase” with one another. This subtle distinction creates a mechanism leading to complex dynamical behaviour not seen in the injection or feedback scenarios.

In this study we differentiate between low and high coupling between the lasers. For weakly coupled lasers, the characteristics of the semiconductor laser is expected to dominate the dynamics. Conversely, for strongly coupled lasers, the simple two laser network is anticipated to influence more. Much of this general discussion is made more robust in Sec. 2.0 with the introduction of the laser rate equations. Lastly the motivation for studying this laser scenario has changed. As a second laser is unlikely to arise in a face-to-face configuration by accident, any pretense that the study of the dynamics is motivated by a quest to afford instabilities is no longer valid. The dynamics is either studied for its own right or the possibility that applications may be found.
1.1 High Coupling

A system of two mutually coupled semiconductor lasers is a simple example of interacting nonlinear oscillators. This system has been experimentally realised using for example edge emitting [47], quantum dot [46], VCSEL [32] and DBR [117] lasers and the observed dynamical phenomena include leader-lag dynamics [47], frequency locking and intensity pulsations [126], and bubbling [30, 111]. For recent reviews on the rich dynamical features of coupled semiconductor lasers we refer to [71, 104].

In this thesis we study this problem theoretically on the basis of a well established rate equation model. In general, there are two time scales which govern the character of the dynamics in coupled semiconductor lasers, namely the period of the relaxation oscillation $T_R$ and the delay time $\tau$ due to the separation between the two lasers. In the long delay case $\tau \gg T_R$, it is well known that the synchronous state of two coupled lasers is in general not stable [118], which gives rise to leader-lag dynamics and noise induced switching between the asymmetric states [77]. In view of applications in small-scale and high-speed devices, we are however interested in the opposite limit, where the delay time is comparable to or even much smaller than the relaxation period. In this limit the rate equation approach is justified if the distance between the lasers is significantly larger than the optical wavelength itself. At even shorter distances, composite cavity models have recently been used to describe effects due to evanescent lateral coupling [28, 10]. Using typical values for the relaxation oscillation in the GHz regime and an optical wavelength of around 1 $\mu$m, for the high coupling study of this thesis, we therefore focus on the case of two coupled lasers at distances between 10 $\mu$m and 500 $\mu$m.

We observe that for small spatial separation and strong optical coupling, the two lasers can mutually lock into one of two stable one-colour states. These states spontaneously break the original symmetry of exchanging the two lasers, and both lasers emit light at precisely the same frequency but at different intensities. In this bi-stable regime the current state of the system can be conveniently observed optically from the amplitude of the light output of the lasers. There also exists a region of hysteresis, where the symmetric one-colour state is jointly stable with the two symmetry broken one-colour states, thus giving rise to a parameter region of tri-stability. For more moderate coupling strength, the symmetry broken one-colour states become unstable, and instead symmetry broken two-colour states appear, which are similar to the ones observed numerically in [98]. In this case, both lasers emit light at the same two optical frequencies, however the intensities of the respective colours are not identical in both lasers.

In order to study these and similar transitions in the dynamics systematically, we introduce a reduced five dimensional model, which allows for a bifurcation analysis using the continuation software AUTO [22]. The transition from a one-colour to a two-colour symmetry broken state then corresponds to a Hopf bifurcation from a symmetry-broken fixed-point state to a symmetry-broken limit cycle state in the language of bifurcation theory. We are in particular interested in the fundamental bifurcations, which bound the domain of symmetry broken two-colour states. We identify the relevant codimension two points, which organise the bifurcation scenario in this region, and explain the mechanism which gives rise to the large region of bi-stability between symmetry-broken two-colour states.
1.2 Low Coupling

The semiconductor laser [9] and CO₂ gas laser [91] share the same dynamical system’s categorisation [113]. To model them requires at least the consideration of an electric field and population inversion [67, 76]. However, there is a mismatch of several orders of magnitude between the lifetime of the photon and that of the carrier. A laser is thus a paradigmatic example of a slow-fast system [62]. Because of this, there exists a special type of periodic orbit whose phase curves have both slow and fast segments [38]. These are the well-known relaxation oscillations of a laser. Intuitively they are the damped oscillations in which a system out of rest settles to its steady state. It’s for laser configuration scenarios only slightly perturbed from the free-running case that the relaxation oscillation frequency is expected to play a large role. The low coupling sections of this thesis examines the interaction between two weakly coupled face-to-face semi-conductor lasers.

Perturbation methods have a long and distinguished history in science. In the past, before the age of the electric calculator and computer, it was the de-facto standard for obtaining numerical solutions to difficult problems. In dynamical systems, much modern research continues using singular perturbation techniques [29, 62]. In this work, we use a perturbation method to introduce coupling between two free-running semiconductor lasers. The “on” and “off” uncoupled states for both lasers are perturbed with the small coupling parameter from which one-colour states emerge. These coincide with the well studied single frequency constant amplitude coupled laser states [128] called Compound Laser Modes (CLMs) in Ref. [27].

Two mutually coupled identical lasers are one of the simplest physical realisations of a two cell network [34, 99]. As such it is important to distinguish the dynamical response due to the nature of the cell and that of the architecture of the network. The equivariant symmetry of being able to exchange the two identical lasers pervades each dynamical state observed. Being invariant with respect to this symmetry or not, allows categorisation of each state into a symmetric and symmetry-broken variant. Our study relies on results and definitions from the literature for equilibria, periodic orbits and quasi-periodic dynamics [81]. The tools open to us are considerable less for categorising chaotic states and so symmetric and symmetry-broken variants are determined by comparing the amplitude of each frequency across the two lasers and comparing to embedded islands of non-chaotic dynamics.

As it is of primary interest for the design of applications, the extent of each dynamical state in terms of the parameter space of the coupling phase and the coupling strength is sought. A qualitative change in the dynamical behaviour of a system is called a bifurcation. As such they occur at the boundary between different dynamics. Bifurcation boundaries are therefore of fundamental importance to understand the mechanism that create each dynamical state discussed and can be used to properly classify distinct dynamics that is not available to experimental investigations. The maximal Lyapunov exponent is determined to ascertain the sensitivity on initial conditions, which is used to delimit parameters choices with chaotic dynamics.
1.3 Applications

One reason the subject of laser dynamics is of broad interest is the possibility that it may lead to real life technological efficiencies and advancements. Currently a technology push is underway to show photonics can replace electronics in the design of computational components, particularly aimed towards high end computing [17]. Delay coupled lasers were highlighted for their possible use in applications [23, 104]. Of particular interest would be sufficiently complex dynamical states that persist as the delay-time goes to zero. Devices based upon these dynamics would likely be faster as the laser light has less distance to travel between the lasers, but more importantly would be open to as much miniaturisation as is technologically possible. Photonic Integrated Circuits (PIC) have allowed the physical realisation and experimental investigation of the dynamics of mutually coupled lasers with ultrashort delay [127, 117, 18, 69]. This year, Pat. [93] reports a significant engineering achievement for the fabrication of cheaper and lower power photonic devices using standard chip-making processes. All this raises the question, can the theoretically discovered dynamics of this thesis for two mutually coupled lasers in the closely coupled limit be applied to the design of photonic devices?

From the technological side there currently exits a strong interest in the development of small-scale devices which are capable of all-optical signal processing. One particular challenge, which has attracted a significant amount of recent research activities, is the design of all-optical memory elements [50, 89, 70, 19, 48, 82, 92]. The goal is to design fast and efficient memory elements which can be switched between at least two different stable states via an external optical signal. At the same time it is also desired that the state of the memory element is accessible optically, and the optical output from one memory element should be able to trigger the switching in further elements, with little or no intermediate processing. Based on these criteria, the question arises whether all-optical memory elements can be realised using two mutually interacting identical single-mode lasers. After having identified a number of interesting regions of multi-stabilities involving one-colour and two-colour states, we numerically demonstrate that these states can indeed be exploited for the design of all-optical memory units. By injecting suitable pulses of light into the coupled lasers we are able to switch between different multi-stable states. In order to assess the technologically relevant speed of the switching events, we define the write time as the minimal injected pulse duration required to trigger the switching event, and the read time as the minimum time after which the state of the memory can be obtained optically. We find read and write times of less than 100ps, which suggests the possibility of fast and simple all-optical memory elements on the basis of identical closely coupled single-mode lasers.

In the future outlook Sec. 7.2 an ultrafast and miniaturisable true random number generator based upon two closely coupled lasers will proposed as an future optoelectrical device.
Figure 1: A light source typically from a master laser enters the cavity of a slave laser. (Single Laser with injection; unidirectional coupling)

Figure 2: A single laser and a mirror are a certain distance $D$ apart. Light re-enters the cavity after a delay $\tau = 2 \left( \frac{D}{c} \right)$. (Lang-Kobayashi Picture; feedback problem)

Figure 3: Two lasers are a certain distance $D$ apart. Light from each enters the cavity of the other after a delay $\tau = \left( \frac{D}{c} \right)$. (mutually-coupled lasers; bidirectionally coupling)
2 The Model

The Maxwell-Bloch equations could be used to model a laser. But a significant reduction in complexity would be highly desirable, especially one which contains the minimum essential elements to describe the phenomena of the system. A partial differential approach exists which includes spatial effects [49]. For studies focused on the temporal evolution of the laser, a system of rate equations with a variable representing the slowly varying complex electric field $E$, the complex polarisation $\mathcal{P}$ and the population inversions $N$ was developed [42]. Following [67], they can be written as a coupled system of two damped harmonic oscillators

$$
\begin{align*}
\frac{d}{dt} + \Lambda_E - i\Omega_E \right] E &= -i\mathcal{P} \\
\frac{d}{dt} + \Lambda_P - i\Omega_P \right] \mathcal{P} &= -iNE \\
\frac{d}{dt} + \Lambda_N \right] N &= P - i (E\mathcal{P}^* - \mathcal{P}E^*)
\end{align*}
$$

where $\Lambda\{E,P,N\}$ are the damping and $\Omega\{E,P\}$ are the oscillations in the fields. $P$ is the pumping rate. The equation has five real dimensions making complex dynamics such as period-doubling, torus bifurcations and chaos possible for a laser even in the absence of external factors. A dynamical systems categorisation was introduced in [7] which discriminates lasers based upon the relative lifetimes of the photon, polarisation and inversions and can be used to reduce the number of variables needed to describe most lasers. Class B lasers, which is the focus of this thesis and includes solid state and semiconductor lasers, need only consider the variables of complex electric field $E$ and carrier density $N$ as $\Lambda_P \gg \Lambda_N, \lambda_E$ and therefore $\mathcal{P}$ is eliminated from Eq. (2.1) as an adiabatic follower [86]. In an effort to choose the simplest description that captures the essence of model without additional complications, the system of equations is often written as

$$
\begin{align*}
\frac{d}{dt} E &= \left(\frac{1 + i\alpha}{1 + i\omega}\right) NE + \kappa f(t) \\
T \frac{d}{dt} N &= P - N - (2N + 1) |E|^2
\end{align*}
$$

where $f(t) = \begin{cases} e^{i\omega_\Delta t} & \text{Injection} \\
E(t - \tau) & \text{Feedback} \\
E_2(t - \tau) & \text{Mutual Coupling}
\end{cases}$

Here a linear gain term suffices and as time is written in order of the photon lifetime, a large parameter $T$ is introduced. This constitutes a significant reduction in complexity for class B lasers, Eqs. (2.2a),(2.2b). As the system is now in effect two dimensional, the possible dynamics are heavily restricted and a class B laser is relatively well-behaved in isolation.
A laser in the real world is naturally subject to stray light incident on its facet from miscellaneous sources. As noted in Ref. [113], this increases the dimensions of the system by at least one returning the possibility that "instabilities" and other rich dynamical behaviour may ensue. To understand what effect this may have, a scenario schematised in Fig. 1 is studied. The model (2.2) modified with the addition of the term Eq. (2.2c) [119] has shown remarkable agreement with experiment [121, 90]. Here $\omega_\Delta$ is the angular frequency of the external source relative to the free-running frequency of laser 1. This scenario has revealed many interesting dynamical phenomena including: phase locking, frequency locking, chirp, frequency oscillations, q-switching [113] and chaos [120]. After many decades of research, the bifurcation picture for single mode and dual mode [83] quantum well laser is relatively complete [120, 88, 11]. Detailed knowledge of the dynamics, their bifurcation boundaries, and regions of multi-stabilities for a laser diode subject to external injection, has allowed new and novel applications to be proposed [89].

A second type of disruption of the laser due to feedback of the laser's own light cannot be understood with the external injection scenario. Essentially this is because the addition of a delay which is the time it takes for the laser after emitting a photon to reabsorb it. Therefore the phase space needed to understand the problem is no longer three dimensional but infinite dimensional. A theoretical and experimental setup schematised in Fig. 2 is considered. A model (2.2) modified with the addition of the term $\Delta t$ and the equation for the electric field and populations inversions in model (2.2) modified with the addition of the term Eq. (2.2e) needs to be doubled to account for a second laser. This results in a system of order six and as delay is also in this problem requires an infinite dimensional phase space to understand it completely.

To modify the model (2.2) for the free-running laser ($\kappa = 0$), each of three scenarios ($\kappa \neq 0$) has a second term added (Eq. 2.2c-2.2e) to Eq. (2.2a). This allows us to loosely differentiate between the dynamic of the lasers when the first term is large compared to the second term and vice versa. For example, for the case of mutually coupled lasers (Eq. 2.2e), the low coupling regime is defined when $\kappa$ in Eq. (2.2) is small, and the characteristics of the class B lasers is expected to dominate the dynamics. Conversely, for high coupling when $\kappa$ in Eq. (2.2) is sufficiently large, the influence of the simple two laser network plays the primary role. Lastly, for medium coupling the two terms are of a similar order of magnitude and a more complex interplay between the effects of the network and specifics of the lasers are expected. This distinction will be useful in for the research work presented in this dissertation and two dynamical system studies for low and high coupling strengths are quasi-independent. Medium coupling is discussed in Sec. 7.1.
2.1 Delay-Differential Description

\[ D = c\tau = \left( j + \frac{C_p}{2\pi} \right) \lambda_0 \]

Figure 4: Schematic diagram of two coupled lasers of wavelength \( \lambda_0 \) separated by a distance \( D \), where \( C_p \in [0, 2\pi) \) and \( j \in \mathbb{N} \).

Two identical single-mode semiconductor lasers with free-running wavelength \( \lambda_0 \) are placed in a face-to-face alignment with a separation \( D \) as sketched in Fig. 4. They are mutually coupled via a certain amount of the light of each entering the other cavity after a delay \( \tau = D/c \), where \( c \) denotes the speed of light. This scenario has been successfully studied in the literature on the basis of rate equation models [63, 51, 75, 98, 128, 27]. In this work we use the following set of delay differential equations:

\[
\begin{align*}
\dot{N}_1(t) &= (1 + i\alpha) N_1(t) E_1(t) + ke^{-iC_p}E_2(t - \tau) \\
\dot{N}_2(t) &= (1 + i\alpha) N_2(t) E_2(t) + ke^{-iC_p}E_1(t - \tau) \\
\dot{E}_1(t) &= \frac{1}{2} \left[ P - N_1(t) - (1 + 2N_1(t)) |E_1(t)|^2 \right] \\
\dot{E}_2(t) &= \frac{1}{2} \left[ P - N_2(t) - (1 + 2N_2(t)) |E_2(t)|^2 \right]
\end{align*}
\] (2.3a, 2.3b, 2.3c, 2.3d)

The parameters include \( P = 0.23 \), the pumping of electron-hole pairs into each laser; \( \alpha = 2.6 \), the line-width enhancement factor; \( T = 392 \), the ratio of the photonic and carrier lifetimes; \( \tau = \{0, 0.2, 20\} \) the dimensionless delay time between the lasers. The coupling strength \( k \) and the coupling phase \( C_p = 2\pi(D \mod \lambda_0) \) are the two main bifurcation parameters in subsequent sections and chapters. The model (2.3) is dimensionless with time measured in units of the photon lifetime \( \tau_p = 1.0204 \) ps. Parameter values were chosen to compare with studies [27, 128] and correspond to a semi-transparent laser.

The dynamical variables \( N_1 \) and \( N_2 \) denote the population inversions, and \( E_1 \) and \( E_2 \) are the slowly varying complex optical fields in laser 1 and laser 2. The rapidly oscillating physical fields can be recovered via \( \hat{E}_{1,2} = E_{1,2}(t)e^{i\Omega_0 t} \), where the optical angular frequency is given by \( \Omega_0 = 2\pi c/\lambda_0 \). The phase factor \( e^{-iC_p} \) in system (2.3) is therefore a consequence of expressing the time delayed physical fields using slowly varying fields via \( \hat{E}_{1,2}(t - \tau)e^{-i\Omega_0 \tau} = e^{-iC_p}E_{1,2}(t - \tau) \).

Any solution of equations (2.3) can be multiplied by a common phase factor in both optical fields leading to a \( S^1 \) symmetry. In addition, as the two lasers are identical, a \( \mathbb{Z}_2 \) symmetry exists due to the ability to swap the lasers. Mathematically these two phase space symmetries can be formulated as [27],

\[
\begin{align*}
(E_1, E_2) &\rightarrow (e^{ib}E_1, e^{ib}E_2), \quad b \in [0, 2\pi) \quad S^1 \text{ symmetry} \\
(E_1, E_2, N_1, N_2) &\rightarrow (E_2, E_1, N_2, N_1) \quad \mathbb{Z}_2 \text{ symmetry}
\end{align*}
\] (2.4)

Both the \( S^1 \) and \( \mathbb{Z}_2 \) symmetries are frequently used and referred to in subsequent sections. For a review of this model see [104].
2.2 Initial Coupling

A natural place to begin a study on mutually coupled lasers is to consider the uncoupled case; \( \kappa = 0 \) in Eqs. (2.3a)(2.3b). A perturbation method considering a constant wave state of fixed carrier density \( N_{1,2} \), amplitude \( A_{1,2} \), frequency \( \omega \) with an allowed phase difference \( \delta \) between the lasers is used to introduce coupling. A small difficulty needs to be overcome as the phase difference between uncoupled lasers is arbitrary. Eqs. (2.3c)(2.3d) can be used to simplify the technique giving the following involution.

\[
A_{1,2}^2 = \frac{P - N_{1,2}}{1 + 2N_{1,2}} \tag{2.5}
\]

The above equation is plotted in Fig. 5 which shows an injective relation between the carrier density \( N_{1,2} \) and intensity \( A_{1,2}^2 \) and is continuous for physically relevant solutions, shown by the solid red curve. Therefore we need only expand in terms of the four constants \( N_{1,2}, \omega \) and \( \delta \). The carrier density for both lasers are expanded as

\[
N_{1,2} = \kappa N_{1,2}^{(0)} + \kappa^2 N_{1,2}^{(2)} + \kappa^3 N_{1,2}^{(3)} + \mathcal{O}(\kappa^4) \tag{2.6}
\]

Specifics of the perturbation method is left to the App. 2.A where the expansion of the phase difference \( \delta \) and frequency \( \omega \) are obtained. The zeroth order term \( \mathcal{O}(1) \) for the magnitude of Eqs. (2.3a)(2.3b) defines the free running laser states.

\[
\begin{align*}
(i) & \quad (N_{1,2}^{(0)})^2(1 + \alpha^2) - 2\omega_0 N_{1,2}^{(0)} + \omega_0^2 = 0 \\
(ii) & \quad (P - N_{1,2}^{(0)}) (2N_{2,1}^{(0)} + 1) = 0 \\
(iii) & \quad (N_{1,2}^{(0)})^2(1 + \alpha^2) - 2\omega_0 N_{1,2}^{(0)} + \omega_0^2 = 0
\end{align*} \tag{2.7}
\]

Here \( \omega_0 \) is the progenitor frequency of the uncoupled but identical lasers. Phase oscillators are known to phase lock at vanishingly small coupling [33].

“on-on” states are obtained by considering \( (i) = 0 \) in Eq. (2.7) where both lasers are in an initial on free-running laser state, i.e. \( N_{1,2}^{(0)} = N_{2,1}^{(0)} = 0 \), sharing frequency \( \omega_0 = 0 \). The on uncoupled laser state is stable for positive pumping, \( P > 0 \) and therefore in the absence of bifurcation the on-on state is expected to be stable. Details are in App. 2.A where the frequency is approximated and phase difference is resolved to be \( \delta = (0, \pi) \). The inversions obtained are

\[
N_{1,2} = \mp \kappa \cos(C_p) \pm \sqrt{2} (1 + \alpha^2) \frac{\tau}{\kappa} \sin(C_p) \sin(a) - \frac{\kappa^3 \tau^2}{2} (1 + \alpha^2) \left[ \sin(2a) \sin(C_p) + \sin^2(a) \cos(C_p) \right] + \mathcal{O}(\kappa^4) \tag{2.8}
\]

where \( a = C_p + \arctan(\alpha) \). The top (-) Eq. (2.8) are the in-phase \( \delta = 0 \) synchronous states, well known to the literature and studied in Refs. [128, 27]. An integration of the full set of Eqs. (2.3) minus the approximation Eq. (2.8) is provided in Fig. 6. The numerics and approximation converge for small coupling strength. The red curve which is the first order term of Eq.(2.8), equivalent to the instantaneous limit \( \tau = 0 \) [128], is within 0.05% at \( \kappa = 0.005 \). Higher order terms are considerably better at low coupling. The inversions, intensity and frequency for an in-phase on-on state at weak coupling strength is provided in Fig. 7.

For this study the “on-on” state with positive pushed inversions will be the more important. For coupling phases \( 0 < C_p < \pi \) this is the anti-phase “on-on” state, \( \delta = \pi \) which is the bottom (+) Eq. (2.8). A standard integration on the full set of Eqs. (2.3) provided in Fig. 8 confirms that the anti-phase state is indeed stable for very weak coupling. As reported in [128] a bi-stability
exists between the two on-on states whose mechanism will be discussed in Sec. 4.2. We note that it is unusual to have a laser locked to an injected signal at positive \( N \) [20], however this appears standard for mutually coupled lasers at very weak coupling. This effect becomes more pronounced as the lasers are brought closer together. “off-off” states are obtained by considering (ii) = 0 in Eq. (2.7) where both lasers are in an initial off-free-running laser state i.e. \( N_1^{(0)} = N_2^{(0)} = P \). It is well known to be stable for negative pumping parameter \( P < 0 \) and as it does not perturb with \( \kappa \) it persists for all regions of our study. It is easy to see however that in the absence of delay \( \tau = 0 \), the off-off state connects with the anti-phase on-on state for \( \kappa = P \) at \( C_p = 0 \). Both states are unstable. “inf-inf” states are obtained by considering (iii) = 0 in Eq. (2.7) where both lasers have an initial free-running laser inversions given by \( N_1^{(0)} = N_2^{(0)} = -1/2 \). This laser state derives its name from the laser having infinite amplitude when the carrier densities are equal to minus a half as shown in Fig. 5. It does not perturb with \( \kappa \). Additionally it is easy to see that in the absence of delay \( \tau = 0 \) the in-phase on-on state connects, without bifurcation, to the inf-inf for \( \kappa = 0.5 \) at \( C_p = 0 \). This is a stable transition but we stress that it almost certainly is a nonphysical artefact of the model.

“on-off” states are obtained by considering the two lasers in different initial free-running laser states. Laser one is in the on state (i) = 0 for relation one and laser two is in the off state (ii) = 0 for relation two of Eq. (2.7). In App. 2.A we obtain the following expansion

\[
N_1 = \kappa \cos(\theta) + \frac{\kappa^2}{4P(1+\alpha^2)} \left[ \frac{2 + \alpha \sqrt{1 + \alpha^2} \sin(2\theta)}{\alpha \cos(\theta) - \sin(\theta)} \right] - \frac{\kappa^2 \tau \sqrt{1 + \alpha^2}}{\alpha \cos(\theta) - \sin(\theta)} + O(\kappa^3)
\]

\[
N_2 = P + \kappa^3 \frac{2P + 1}{P^2(1+\alpha^2)} \cos(\theta) + O(\kappa^4)
\]

where \( \theta = \arctan(\alpha) - 2C_p \). As laser two is in the off state which is unstable for positive pumping \( P > 0 \), the on-off state is expected to be unstable, without bifurcation, at initial coupling. Indeed for non-identical lasers or if laser two had pumping below threshold the converse would be true. It will be shown in Sec. 3.1 that on-off states were stable for very high coupling. This occurs after the state gains stability via a Hopf bifurcation. The approximation (2.9) gives a reasonable estimate despite large \( \kappa \). Via continuation software [22] it is confirmed that the unstable mixed-phase compound laser modes of [27] and the unstable asynchronous states of [128] owe their root to the on-off free-running laser state, Fig. 9. It’s noted however, due to a turning around of solutions, the linear perturbation method is unable to give a reasonable estimate for the fixed constants, \( N_{1,2}, A_{1,2}, \omega \) and \( \delta \).

For all states discussed above, the delay time plays very little quantitative role at weak coupling strengths appearing in \( O(\kappa^2) \) and higher order terms. The frequencies of each delay-created on-on states were determined in Ref. [27] by the following transcendental equation

\[
\omega_A = \mp \kappa(1 + \alpha^2) \frac{\sin(\omega_A \tau + \tan^{-1} \alpha + C_p)}{\sqrt{1 + \alpha^2}}
\]

where the top (-) are for in-phase and the bottom (+) are for anti-phase respectively. Solutions of Eq. (3.2) are graphically shown in Fig. 10. The plot illustrates that for any fixed delay time \( \tau \), a small enough \( \kappa \) may be chosen so that a single in-phase (anti-phase) on-on state exists i.e. no
delay-created one-colour states. This defines the weakly coupled limit which is analytically given by the following inequality [124, 128]

\[
\kappa < \frac{1}{\tau (1 + \alpha^2) \delta}
\]  

(2.11)

For \( \tau = 20 \) and \( \alpha = 2.6 \), \( \kappa \) less than 0.0179 satisfies this condition. Within this limit the multi-valued functions for the amplitudes, inversions and frequencies Eq. (3.2) become single valued. Outside this limit, as seen in Ref. [27] a saddle-node bifurcation complicates the dynamics. In this low coupling study, we restrict \( \kappa \) to well below this limit, up to a maximum of 0.005, and argue that therefore an analytical and bifurcation explanation in the instantaneous limit \( \tau = 0 \) suffices.

The angular relaxation oscillation frequency for anti-phase on-off states at \( \tau = 0 \) is approximated to the order of \( O(1/\tau) \), where \( T \) is a large parameter, by

\[
\omega_{\tau} \approx \left\{ \frac{2 (P + \kappa \cos (C_p))}{T} \right\}^{\frac{1}{2}}
\]  

(2.12)

This expression forms part of a side calculation in App. 3.D. Physically interpreted they are the damped oscillations in which a system perturbed from a steady state returns to that state. For parameters chosen in this thesis, this corresponds to a free-running relaxation oscillation frequency of 5.34 GHz. Maximising the coupling term within this study’s parameter range changes this value by at most 1%.

The above discussion purposely chose not to mention a conjugate state to the “on-off” state, namely: the “off-on” state. These states occur due to the \( Z_2 \) symmetry Eq. (2.4) of being able to exchange the two identical lasers. This has an important influence on the dynamics. Following Refs. [79, 80] we observe that its useful to decompose the phase space \( \mathbb{R}^n = X^+ \oplus X^- \) where the action of the symmetry group \( R \) is the identity \( Rx = x \) for \( x \in X^+ \) and the central reflection \( Rx = -x \) for \( x \in X^- \). In the case of two mutually coupled lasers, a natural coordinate representation which achieves this are the sum “S” and difference “D” of the phases \( \phi_{1,2} \), amplitudes \( A_{1,2} \) and inversions \( N_{1,2} \) of the lasers.

\[
X^+ = \begin{pmatrix} \phi_S \\ A_S \\ N_S \end{pmatrix}, \quad X^- = \begin{pmatrix} \phi_D \\ A_D \\ N_D \end{pmatrix}
\]  

(2.13)

Another phase space symmetry exists in the system. It is common to reduce the order of the system (2.3) using the \( S^1 \) symmetry Eq. (2.4) as done in [128, 27]. Graphically this is illustrated in Fig. 11. The electric fields of laser one and two are rotated around the origin freely by the symmetry showing that the absolute (sum) phase doesn’t matter only the phase difference. This has the effect of making on-on and on-off constant wave states equilibria which helps make analysis substantially easier. In the language of [81, 64], on-on, off-off and inf-inf are fixed or F-type equilibria. We will refer to these as the three symmetric states as they are invariant with respect to the \( Z_2 \) symmetry Eq. (2.4). For on-on states, the term symmetric one-colour states is used in succeeding sections. The on-off is of mirror or M-type with the off-on and on-off R-conjugate to one another. Anything effecting one of these states are automatically mirrored in the other state. We refer to the symmetry-broken one-colour states. As will be seen in succeeding sections, the \( Z_2 \) symmetry affects limit cycles, quasi-periodic, chaotic dynamics and the bifurcations between them. Terminology will be borrowed from the literature as needed.
2.3 Reduced Coordinate System

Our aim is to understand the bifurcation structure associated with the various one-colour states and two-colour states presented in the previous sections. As we are mostly interested in the closely coupled limit, we now formally set \( \tau = 0 \). System (2.3) then becomes a six dimensional system of ordinary differential equations.

The system still possesses the \( S^1 \) symmetry (2.4) which can be exploited in order to reduce the number of dimensions of the system. This is graphically demonstrated in Fig. 11. One popular way of achieving this is by rewriting the system (2.3) with the dynamical variables \((|E_1|, |E_2|, \phi_D, \phi_A, N_1, N_2)\) using \( E_{1,2} = |E_{1,2}| e^{(\phi_A + (\phi_D/2))} \). Then the dynamical variable for the absolute phase \( \phi_A \) decouples from the rest of the system and we are left with a five dimensional system. Although widely used, this approach is problematic, because the dynamical variable for the phase difference between the electric fields \( \phi_D \) is not well defined if either \( E_1 \) or \( E_2 \) vanishes. As a consequence, \( \phi_D \) can jump discontinuously as one of the electric fields goes through the origin. This is schematically demonstrated in Fig. 12. To see this mathematically consider the differential equation for \( \phi_D \) which is given by

\[
\dot{\phi}_D = \alpha (N_2 - N_1) + \kappa \left( \frac{|E_2|}{|E_1|} \sin(C_p - \phi_D) - \frac{|E_1|}{|E_2|} \sin(C_p + \phi_D) \right)
\]

The discontinuity in \( \phi_D \) manifests itself in the form of singularities at \( |E_{1/2}| = 0 \), which make it difficult to use numeric continuation software to explore the dynamical features of the system.

In order to avoid these singularities, we introduce a five dimensional coordinate system \((q_x, q_y, q_z, N_1, N_2)\), where the variables are defined via

\[
\begin{align*}
q_x + iq_y &= 2E_1^*E_2 \\
q_z &= |E_1|^2 - |E_2|^2
\end{align*}
\]  

(2.14a) \hspace{1cm} (2.14b)

These equations are graphically justified in Fig. 13. The multiplication by 2 in (2.14a) ensures that the Euclidean length of the \( q \)-vector equals the total intensity output of both lasers, \( R = (q_x^2 + q_y^2 + q_z^2)^{1/2} = |E_1|^2 + |E_2|^2 \). The coordinates \( q_x, q_y, q_z \) are analogous to the usual Poincaré sphere representation of polarised light, and per definition do not depend on the absolute phase \( \phi_A \). Therefore the new variables are invariant under the \( S^1 \) symmetry of the original system (2.4). The \( \mathbb{Z}_2 \) symmetry now operates as follows

\[
(q_x, q_y, q_z, N_1, N_2) \rightarrow (q_x, -q_y, -q_z, N_2, N_1)
\]  

(2.15)

Rewriting the system (2.3) for \( \tau = 0 \) in terms of the new coordinates (2.14) yields the five dimensional system

\[
\begin{align*}
\dot{q}_x &= q_x (N_1 + N_2) + \alpha q_y (N_1 - N_2) + 2\kappa R \cos(C_p) \\
\dot{q}_y &= q_y (N_1 + N_2) - \alpha q_x (N_1 - N_2) - 2\kappa q_z \sin(C_p) \\
\dot{q}_z &= q_z (N_1 + N_2) + R (N_1 - N_2) + 2\kappa q_y \sin(C_p) \\
T \dot{N}_1 &= P - N_1 - (1 + 2N_1) (R + q_z)/2 \\
T \dot{N}_2 &= P - N_2 - (1 + 2N_2) (R - q_z)/2.
\end{align*}
\]  

(2.16)

As intended, there are now no singularities in the dynamical variables, and the dynamical equations are invariant under the \( \mathbb{Z}_2 \) symmetry operation (2.15). It is this reduced system that will be used for bifurcation analysis in the next section.
Figure 5: Graph showing solutions to Eq. (2.5) for $P = 0.23$. Black dots are at the free-running off-state $(0.23, 0, 0)$ and the free-running on-state $(0, 0, 0.23)$. Solid line shows the bounded range of physical relevant solutions of two coupled lasers.

Figure 6: A numerical integration sweep for system (2.3) minus the in-phase symmetric one-colour expansion approximation (2.8) at $\tau = 20$, $C_p = 0.1\pi$ for small $\kappa$ is shown.
Figure 7: Timetraces and spectra showing the dynamics at $\tau = 20$, $\kappa = 0.0005$, $C_p = 0.05\pi$. (in-phase one-colour state) Inset highlights the in-phase dynamics between the two lasers.
Figure 8: Timetraces and spectra showing the dynamics at $\tau = 20$, $\kappa = 0.0005$, $C_p = 0.4\pi$. (anti-phase one-colour state) Inset highlights the anti-phase dynamics between the two lasers.
Figure 9: Following the on-on and on-off states for $\tau = 0$ and $C_p = 0.275\pi$ by increasing the coupling strength. Dashed lines are unstable and solid lines are stable solutions. Points: “H” are Hopf bifurcations, “P” pitchfork bifurcations, and “SN” are saddle node bifurcations.
Figure 10: Graphical solution of Eq. (3.2) for the in-phase case. The LHS side is represented by the black straight line; the blue and red sine waves are the RHS of the equation at different delay times. Solutions are where the two lines cross.

Figure 11: The action of the $S^1$ symmetry (2.4) is applied to the electric fields, graphically illustrating that the absolute argument of them doesn’t matter.
Figure 12: Demonstration of the discontinuity of $\phi_D$ at the origin. A small change in the electric field $E_1$ of the first laser can lead to a large change in the polar coordinate $\phi_D$ to $\phi_D + \pi$.

Figure 13: The new coordinate set (2.14a) (2.14b) encodes the relative argument and vector lengths of the electric fields.
Chapter Appendix

2. A Perturbation method

Solutions of the form obeying the one colour ansatz $E_1 = A_1 \exp[i\omega t], E_2 = A_2 \exp[i\omega t] \exp[i\delta]$ are sought. Considering Eqs. (2.3c) (2.3d) the involutory condition Eq. (2.5) between the amplitude $A$ and inversion $N$ for each laser is obtained. This is plotted in Fig. 5 and shows a 1-1 relation between the inversion $N$ and intensity $A^2$ for physically relevant solutions, shown by the solid red curve.

Considering the magnitude of (2.3a)(2.3b) the following two equations are obtained which relate the inversions $N$, the amplitude $A$ and the frequency $\omega$ for both lasers.

$$[N_{1,2}^2(1 + \alpha^2) - 2\omega N_{1,2} + \omega^2] A_{1,2}^2 = \kappa^2 A_{2,1}^2$$  \hspace{1cm} (2.17)

The amplitudes $A_{1,2}$ in Eq. (2.17) are eliminated by using the involution Eq. (2.5) and thereby the following two relations between $N_1, N_2$ and $\omega$ are obtained.

$$(2N_{2,1} + 1)(P - N_{1,2}) \left[ N_{1,2}^2(1 + \alpha^2) - 2\omega N_{1,2} + \omega^2 \right] = \kappa^2(P - N_{2,1})(2N_{1,2} + 1)$$  \hspace{1cm} (2.18)

The argument of (2.3a)(2.3b) yields two relations which relate the phase difference $\delta$, the inversions $N$ and the frequency $\omega$ for $\kappa \neq 0$ and $A_{1,2} \neq 0$.

$$[\omega - \alpha N_{1,2}] \cos(\pm \delta - C_p - \omega \tau) = N_{1,2} \sin(\pm \delta - C_p - \omega \tau)$$  \hspace{1cm} (2.19)

For $\kappa = 0$ one-colour states with arbitrary $\delta$ exist. With no need to consider the amplitudes $A_{1,2}$, greatly simplifies the perturbation method for small parameter $\kappa$ on the two double relations Eqs. (2.18) and (2.19).

The four constants $N_1, N_2, \omega$, and $\delta$ are expanded as part of perturbation technique. For the purposes of this study, it suffices to resolve $\omega$, and $\delta$ to the third order or less. Therefore, in addition to Eq. (2.6),

$$\omega = \omega_0 + \kappa \omega_1 + \kappa^2 \omega_2 + \mathcal{O}(\kappa^3)$$  \hspace{1cm} (2.20)

$$\delta = \delta_1 + \kappa \delta_2 + \kappa^2 \delta_3 + \mathcal{O}(\kappa^3)$$  \hspace{1cm} (2.21)

The perturbation method may then be conducted on the four relations (2.18) and (2.19) above.

For Eq. (2.19) the relations were further simplified with the trigonometric identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$  \hspace{1cm} (2.22a)

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$  \hspace{1cm} (2.22b)

and the small angle approximations

$$\sin(\epsilon \theta) \approx \epsilon \theta$$  \hspace{1cm} (2.23a)

$$\cos(\epsilon \theta) \approx 1 - \epsilon^2 (\theta^2/2)$$  \hspace{1cm} (2.23b)
In the following, the perturbation method was conducted on Eqs. (2.18) and (2.19). Eq. (2.7) is the $O(1)$ term of Eq. (2.18) with following equations one, two, three being successive higher order terms. The last three relations relate to the expansion of Eq. (2.19).

$$
\left[ N_{1,2}^{(1)}(2N_{2,1}^{(0)} + 1) - 2(P - N_{1,2}^{(0)} N_{2,1}^{(1)}) \right] \left[ (i) \right] =
$$

$$
\left( 1 + \alpha^2 \right) \left[ N_{1,2}^{(1)} \right]^2 + 2N_{1,2}^{(0)} N_{1,2}^{(2)} - 2\alpha \left[ N_{1,2}^{(2)} \omega_0 + N_{1,2}^{(1)} \omega_1 + N_{1,2}^{(0)} \omega_2 + 2\omega_0 \omega_2 + \omega_1^2 \right]
$$

$$
+ \left[ (i) \right] \left[ 2N_{1,2}^{(2)} \left( P - N_{1,2}^{(0)} - 2N_{1,2}^{(1)} - N_{1,2}^{(2)} \right) \right] \left[ (iii) \right] \left[ (iii) \right] = \left[ (iv) \right] \left[ (v) \right] + \left[ (vi) \right] \left[ (vii) \right] \times \left[ (i) \right]
$$

$$
2PN_{1,2}^{(1)} - 2N_{1,2}^{(0)} N_{1,2}^{(1)} - 2N_{1,2}^{(1)} N_{1,2}^{(0)} =
$$

$$
\left[ (vii) \right] \left[ (vi) \right] + \left[ (iv) \right] \left[ (vi) \right] + 2 \left[ (iii) \right] \left[ (iii) \right] \left[ (i) \right] \left[ \left( 1 + \alpha^2 \right) \left( N_{1,2}^{(0)} N_{1,2}^{(2)} + N_{1,2}^{(1)} N_{1,2}^{(2)} \right) \right]
$$

$$
- \alpha \left[ \omega_0 N_{1,2}^{(1)} + \omega_2 N_{1,2}^{(0)} + \omega_1 N_{1,2}^{(0)} + \omega_0 N_{1,2}^{(1)} \right] + \left( \omega_0 \omega_2 + \omega_1 \omega_2 \right)
$$

$$
+ \left[ 2N_{1,2}^{(3)} \right] - 2 \left( N_{1,2}^{(3)} N_{1,2}^{(0)} + N_{1,2}^{(2)} N_{1,2}^{(1)} + N_{1,2}^{(1)} N_{1,2}^{(2)} + N_{1,2}^{(0)} N_{1,2}^{(3)} \right) - N_{1,2}^{(3)} \right] \left[ (i) \right]
$$

$$
(\omega_0 - \alpha N_{1,2}^{(0)}) \cos(\pm \delta_1 - C_p - \tau \omega_0) = N_{1,2}^{(0)} \sin(\pm \delta_1 - C_p - \tau \omega_0)
$$

$$
\sin(\pm \delta_1 - C_p - \tau \omega_0) \left[ N_{1,2}^{(1)} + (\omega_0 - \alpha N_{1,2}^{(0)}) (\pm \delta_2 - \tau \omega_1) \right] = \cos(\pm \delta_1 - C_p - \tau \omega_0) \left[ \omega_1 + N_{1,2}^{(0)} (\omega_1 \tau - \delta_2) - \alpha N_{1,2}^{(1)} \right]
$$

(2.24)
\[
\begin{align*}
&\sin(\pm \delta_1 - C_p - \tau \omega_0) \times \\
&\left[ N_{1,2}^{(2)} - N_{1,2}^{(0)} \left( \frac{\pm \delta_2 - \tau \omega_1}{2} \right)^2 + (\omega_1 - \alpha N_{1,2}^{(1)})(\pm \delta_2 - \tau \omega_1) + (\delta_3 - \omega_2 \tau)(\omega_0 - \alpha N_{1,2}^{(0)}) \right] \\
&= \cos(\pm \delta_2 - C_p - \tau \omega_0) \times \\
&\left[ (\omega_2 - \alpha N_{1,2}^{(2)}) - N_{1,2}^{(1)}(\pm \delta_2 - \tau \omega_1) - \left( \frac{\pm \delta_2 - \tau \omega_1}{2} \right)^2 (\omega_0 - \alpha N_{1,2}^{(0)}) - N_{1,2}^{(0)}(\delta_3 - \omega_2 \tau) \right]
\end{align*}
\]

Eq. (2.7) allows us to define the three free-running laser states as the “on” state when \(i = 0\) which implies \(N^{(0)} = 0\) and \(\omega_0 = 0\), the “off” state when \(ii = 0\) which implies \(N^{(0)} = P\) and the “inf” state when \(iii = 0\) which implies \(N^{(0)} = -1/2\). This gives nine states to consider, namely “on-on”, “off-off”, “inf-inf”, “on-off”, “off-on”, “on-inf”, “inf-on”, “off-inf” and “inf-off” states. The first three give symmetric states whilst the last six give three symmetry-broken pairs. The last four are forbidden by the reversal of laser indices in terms (ii) and (iii) of Eq. (2.7).

2. A. 1 Symmetric - “on-on”

It is relatively straightforward to show that off-off and inf-inf states do not perturb for the first few terms of their expansions. \(i = 0\) in Eq. (2.7) for both relation one and relation two defines the on-on state and implies \(N^{(0)} = 0\) and \(\omega_0 = 0\). \(ii = P\) and \(iii = 1\). From Eq. (2.24), \((v) = 0\) is satisfied and \((vi) = (2P - 1)N^{(1)}\). Eq. 2.25 implies

\[ 1 = (1 + \alpha^2) \left( N^{(1)} \right)^{1/2} - 2\alpha N^{(1)} \omega_1 + \omega_1^2 \]  

(2.30)

Turning to the expansion terms generated from Eq. 2.19, Eq. 2.27 is satisfied. Relation one and relation two of Eq. (2.28) yield

\[
\begin{align*}
N^{(1)} [\alpha + \tan(\pm \delta_1 - C_p)] &= \omega_1 \\
N^{(1)} [\alpha + \tan(-\delta_1 - C_p)] &= \omega_1 
\end{align*}
\]

(2.31)

(2.32)

Satisfying both these terms simultaneously fixes \(\delta_1 = \{0, \pi\}\). Furthermore using Eq. (2.30), \(N^1 = \cos(\delta_1 - C_p)\) and \(\omega_1 = \sin(\delta_1 - C_p) + \alpha \cos(\delta_1 - C_p)\). Eq. (2.29) gives the relation

\[
N^{(2)} \omega_1 = \omega_2 \cos(\delta_1 - C_p) + \tau \omega_1 
\]

(2.33)

Considering the Eq.(2.26) gives a second relation

\[
\omega_2 \sin(\delta_1 - C_p) + N^{(2)} \omega_1 = 0 
\]

(2.34)

Finally combining conditions (2.34) and (2.33).

Summarising all the results above, we obtain \(\delta = \{0, \pi\}\), the perturbed solution (2.8) and the following expansion for the frequency:

\[
\omega = \mp \kappa \sqrt{1 + \alpha^2 \sin(a)} \\
\pm \kappa^2 (1 + \alpha^2) \tau \cos(a) \sin(a) \\
+ O(\kappa^3)
\]

(2.35)

where \(a = C_p + \arctan(\alpha)\).
2.4.2 Symmetry-broken - “on-off”

Here we consider the solution where the free running states are $N_1^{(0)} = 0$, $N_2^{(0)} = P$ sharing the same progenitor frequency $\omega_0 = 0$. The calculations are more involved than the “on-on” case as the two relations for each equation needs to be considered separately. We summaries the work here.

Relation one of Eq. (2.7) gives (i) $= 0$, (ii) $= P$ and (iii) $= 2P + 1$ which implies $N_1^{(0)} = 0$ and $\omega_0 = 0$. Eq. (2.24) gives (iv) $= -N_1^{(1)}(2P + 1) + 2PN_2^{(1)}$ and (v) $= 0$ is immediately satisfied. Eq. (2.25) says (vi) $= 1$ and therefore

\[
(1 + \alpha^2) \left[ N_1^{(1)} \right]^2 - 2\alpha \omega_1 \left[ N_1^{(1)} \right] + \omega_1^2 = 1 \tag{2.36}
\]

Relation two of Eq. (2.7) gives (ii) $= 0$, (i) $= P^2(1 + \alpha^2)$ and (iii) $= 1$. Eq. (2.24) gives (iv) $= 0$ which implies $N_2^{(1)} = 0$. Eq. (2.25) has (vii) $= 0$ and so $N_2^{(2)} = -2N_1^{(1)}N_2^{(1)}$ which implies $N_2^{(2)} = 0$. Consider relation two of Eq. (2.27) defines $\delta_1 = \arctan(\alpha) - C_p$ which along with Eq. (2.28) gives $\delta_2 = -\omega_1 \left[ 1/(P(1 + \alpha^2)) + \tau \right]$. Relation one of Eq. (2.29) gives the relation

\[
N_1^{(2)} = \omega_2 \cos(\delta_1 - C_p) + \omega_1 \left[ \frac{1}{P(1 + \alpha^2)} + 2\tau \right] \tag{2.37}
\]

Eq. (2.27) for relation one is immediately satisfied while Eq. (2.28) gives

\[
\left[ \tan(\delta_1 - C_p) + \alpha \right] N_1^{(1)} = \omega_1 \tag{2.38}
\]

Combining Eqs (2.36) and (2.38) fixes $N_1^{(1)} = \cos(\delta_1 - C_p)$ and $\omega_1 = \sin(\delta_1 - C_p) + \alpha \cos(\delta_1 - C_p)$. Relation one of Eq. (2.26) gives

\[
\omega_2 \sin(\delta_1 - C_p) = \alpha \omega_1 N_1^{(2)} + \frac{1}{2P} \tag{2.39}
\]

and relation two gives

\[
N_2^{(3)} = \frac{2P + 1}{P^2(1 + \alpha^2)} N_1^{(1)} \tag{2.40}
\]

Finally combining Eqs. (2.37) and (2.39) determines $N_1^{(2)}$ and $\omega_2$. Summarising all the results above, we obtain the perturbed solution (2.6) along with the expansions

\[
\delta = \tan^{-1}(\alpha) - 2C_p + \kappa \left[ \frac{\sin(2C_p)}{P\sqrt{1 + \alpha^2}} + \kappa \sqrt{1 + \alpha^2} \sin(2C_p) + \mathcal{O}(\kappa^2) \right] \tag{2.41}
\]

and

\[
\omega = \kappa \sqrt{1 + \alpha^2} \left[ \sin(2\tan^{-1}(\alpha) - 2C_p) \right] + \mathcal{O}(\kappa^2) \tag{2.42}
\]
3 Dynamics Observed

The purpose of this chapter is to collect together in one place all the dynamical behaviour observed for the two studies. Each dynamical state is presented as having a symmetric and symmetry-broken variant. These are defined rigidly as being invariant with respect to \( \mathbb{Z}_2 \) symmetry (2.4) for one-colour, two-colour and undamped relaxation oscillation states and looser for more complicated dynamics.

One colour states were already introduced in Sec. 2.2 with the focus on initial and low coupling. The two symmetric one colour states, namely in-phase and anti-phase states are the only types stable at weak coupling strengths. The first Sec. 3.1 of this chapter, introduces one-colour states at strong coupling and low separation between the lasers. In this regime, symmetry-broken one-colour are shown to be stable. Section 3.2 of this chapter deals with the two-colour states which exist at more moderate coupling strengths and persist for larger delays between the lasers. The last two Secs. 3.3 and 3.4 introduce undamped relaxation and quasi-periodic dynamics as a feature of weak coupling but possibly large delay.

3.1 One-Colour States

One-colour states, which are also known as compound laser modes (CLMs) [27] are constant amplitude and single frequency solutions to (2.3) whereby the two lasers are frequency locked. They are characterised by the following ansatz,

\[
E_1(t) = A_1 e^{i \omega_A t} \\
E_2(t) = A_2 e^{i \omega_A t} e^{i \delta_A}
\]

(3.1)

with real constants \( A_1, A_2, N_1^c, N_2^c, \delta_A \) and \( \omega_A \). The frequency \( \omega_A \) is the common locked frequency of the slowly varying optical fields of the two lasers and \( \delta_A \) allows for a constant phase difference between them. They emerge from the free-running lasers’ “on” states as seen in Sec. 2.2. The ansatz above can be split into two complementary classes: (i) symmetric CLMs (\( A_1 = A_2 \)) which are invariant under the \( \mathbb{Z}_2 \) symmetry and (ii) symmetry-broken CLMs where both lasers lase at different intensities (\( A_1 \neq A_2 \)). Further details on this classification are given in App. 3.A.

Symmetric CLMs can be further sub-divided into “in-phase” (\( \delta_A = 0 \)) and “anti-phase” (\( \delta_A = \pi \)) solutions and their stability is extensively studied in [27, 128]. In particular, it was found that symmetric CLMs can lose their stability via Hopf, saddle node or pitchfork bifurcations, and the stability boundaries were obtained via numerical continuation techniques [27] or in the instantaneous limit \( \tau = 0 \) analytically by using the characteristic equation of the system [128]. In the literature, symmetry-broken CLMs play a role in the stability analysis of symmetric CLMs, but are themselves not stable, see Fig. 14.

A principal result of this research is that symmetry-broken CLMs are shown to be stable for small delay and relatively high coupling between the lasers. This is demonstrated numerically in the middle and right columns of Fig. 15 where optical field intensities and frequency spectra of two stable symmetry-broken states are plotted. Due to the \( \mathbb{Z}_2 \) symmetry of exchanging the two lasers,
symmetry-broken CLMs always exist in pairs. For purposes of display, a parameter set was chosen where a symmetric CLM is also stable as shown in the left column of Fig. 15, giving rise to a tri-stability in CLMs. In the language of Sec. 2.2 symmetric CLMs are the “on-off” states whilst symmetry-broken CLMs are “on-on” states meaning that they emerge from zero coupling by considering one laser in the “on” state and the other laser in the “off” state. Fig. 9 shows the tri-stability of one-colour states and the connection between in-phase and anti-phase “on-on”, “on-off”, and “off-on” states as the coupling strength is varied.

The frequencies of the CLMs in the bottom panels of Fig. 15, can be analytically determined by plugging the ansatz (3.1) into the system of ODEs (2.3) [27, 128]. The equations for \( \dot{E}_1 \) and \( \dot{E}_2 \) yield the following relation between the frequency \( \omega_A \) of the CLM and the phase difference \( \delta_A \) between the lasers

\[
\frac{\omega_A^2}{\kappa^2 (1 + \alpha^2)} = \sin^2 (C_p + \omega_A \tau + \arctan \alpha) - \sin^2 \delta_A
\]

(3.2)

For in-phase \( \omega_{\text{in}} = \omega_A(\delta_A = 0) \) and anti-phase \( \omega_{\text{ap}} = \omega_A(\delta_A = \pi) \) CLMs, this immediately gives an implicit function for the frequency. For the symmetry-broken one-colour states one additionally needs to consider the fixed point solutions \( N_{1,2} = 0 \) for the inversions. An implicit solution is calculated in App. 3.B for the system with delay time \( \tau \). A secant method is then used to obtain the numerical values appearing as dot-dashed vertical lines in Fig. 15.

### 3.2 Two-Colour States

In this section we introduce two-colour states which are stable for small delay and moderate to low coupling strength between the lasers. We stress that the two-colour states are induced by the coupling between the lasers alone, the uncoupled lasers are single-mode only. Like the one-colour states, the two-colour states can either be symmetric or symmetry-broken with respect to the \( \mathbb{Z}_2 \) symmetry of exchanging the two lasers.

An example of a stable symmetry-broken two-colour state is shown in Fig. 16. Due to the \( \mathbb{Z}_2 \) symmetry of being able to exchange the two lasers, a twin two-colour state is also stable. In the optical spectrum in the bottom panel of Fig. 16 we see that there are indeed only two dominating frequencies \( \omega_A \) and \( \omega_B \) and both lasers lase at these two frequencies, but with unequal intensities. In the time traces of the field amplitudes this gives rise to beating oscillations as shown in the upper left panel of Fig. 16. The corresponding inversions shown in the upper right panel of Fig. 16 also reflects the symmetry-broken nature of this state and in addition shows small oscillations at the beating frequency.

An example for symmetric two-colour states and their connection with symmetric CLMs is shown in Fig. 17. Starting from the in-phase CLM in the left hand panel of Fig. 17 and increasing the coupling phase \( C_p \), we observe a torus bifurcation where the frequency of the anti-phase CLM is turned on. Increasing \( C_p \) further smoothly transfers power from the in-phase mode to the anti-phase mode until a second torus bifurcation kills the in-phase CLM’s frequency and only the anti-phase CLM remains. Frequency spectrum snapshots as the parameter \( C_p \) is changed are shown from left to right in Fig. 17. We note that symmetric two-colour states exist only over very small ranges of the coupling phase \( C_p \), however we will show in Sec. 4.1 that they are crucial for the overall understanding of the bifurcation structure in closely coupled single mode lasers. The symmetric two-colour states can be interpreted as beating between symmetric CLMs. This is similar to the beating between delay-created external cavity modes for a single laser with mirror
[25] which occurs when both ECM states share approximately equal inversions.

A useful ansatz [98] which approximates the dynamics of symmetric and symmetry-broken two-colour states is given by

\[
\begin{align*}
E_1(t) &= A_1 e^{i\omega_A t} + B_1 e^{i\omega_B t}, \\
N_1(t) &= N_{1c}, \\
E_2(t) &= A_2 e^{i\omega_A t} e^{i\delta_A} + B_2 e^{i\omega_B t} e^{i\delta_B}, \\
N_2(t) &= N_{2c},
\end{align*}
\]

with real constants \(A_{1,2}, B_{1,2}, N_{1,2c}, \omega_{A,B}\) and \(\delta_{A,B}\). While this ansatz is a straightforward generalisation of the CLM ansatz (3.1), with a second frequency \(\omega_B\), we stress that in contrast to the CLM ansatz, equations (3.3) fulfill the original system (2.3) only approximately. The presence of two frequencies give rise to oscillations in the intensities of the electric fields of the form

\[
|E_1(t)|^2 = A_1^2 + B_1^2 + 2A_1B_1 \cos ((\omega_A - \omega_B) t)
\]

and similarly for \(|E_2|^2\). According to (2.3) this then also leads to oscillations in the population inversions, as we have seen in the upper left panel of Fig. 16, and thereby contradicts the assumptions of constant \(N_{1,2}(t) = N_{1,2c}\). However, as the parameter \(T\) is large, ansatz (3.3) is often well justified in practice, in particular if the beating frequency \(\omega_A - \omega_B\) is also large. In the same way as for CLMs, the ansatz (3.3) can be split into symmetric \((A_1 = A_2, B_1 = B_2)\) and symmetry-broken solutions.

In the case of symmetric two-colour states, the frequencies \(\omega_A\) and \(\omega_B\) correspond to the frequencies of in-phase and anti-phase CLMs and are obtained from (3.2). For symmetry-broken states the analytical calculation of \(\omega_A\) and \(\omega_B\) is shown in App. 3.C.

### 3.3 Undamped Relaxation Oscillations

Self pulsations in the form of q-switching, undamped relaxation oscillations or mode beating have been widely studied in the context of a laser with feedback [109, 124, 115] and a laser with external injection [113] [120]. They were also observed for two mutually coupled lasers [97, 51]. In this section we show that the basic mechanism for two identical coupled lasers is necessarily different and we introduce symmetric one-colour states with undamped relaxation oscillations as a dominant feature at low coupling strengths.

Oscillatory solutions of system (2.3) are sought by considering a loss of transversal stability of the anti-phase one-colour state. Standard linearization is done for \(\tau = 0\) in App. 3.D. Removing two non-critical eigenvalues, results in the following simplified \(3 \times 3\) matrix.

\[
\begin{pmatrix}
2\kappa \cos(C_p) & -2\kappa \sin(C_p) & \alpha \\
2\kappa \sqrt{P} \sin(C_p) & 2\kappa \cos(C_p) & \sqrt{P} \\
0 & -2\sqrt{P} & -2P + 1
\end{pmatrix}
\begin{pmatrix}
\phi_D \\
A_D \\
N_D
\end{pmatrix}
= i\omega_H
\begin{pmatrix}
\phi_D \\
A_D \\
N_D
\end{pmatrix}
\]

A Routh-Hurwitz type scheme is used to tackle the above eigenvalue-eigenvector problem (3.5) where a pair of purely imaginary eigenvalues \(\omega_H\) is sought.

\[
\omega_H = \frac{\omega_A}{T} + \frac{2P + 1}{1 - 2\kappa \cos(C_p)} \left(1 + \frac{\omega_B}{1 - 2\kappa \cos(C_p)} \right) + 4\kappa^2
\]

(3.6)
This is the angular frequency of any limit cycle born. As can be clearly seen the eigenspace of Eq. (3.5) is in fact the X− subspace (2.13). Hence any periodic orbit born is necessarily of symmetric or S-type [81] and therefore invariant to the following spatiotemporal symmetry:

$$\begin{pmatrix}
\phi_S \\
\phi_D \\
A_S \\
A_D \\
N_S \\
N_D
\end{pmatrix}
\mapsto
\begin{pmatrix}
\phi_S \\
-\phi_D \\
A_S \\
-A_D \\
N_S \\
-N_D
\end{pmatrix}
(t + \frac{\gamma}{2}) \quad \text{S-type cycle} \quad (3.7)
$$

where $\gamma$ is the minimal period of the limit cycle. This restricts the types of bifurcations that can occur [79, 80]. For example, as it is forbidden for Floquet multipliers to cross the unit circle at −1 for S-type cycles of even discrete symmetry group ($\mathbb{Z}_2$); period doubling bifurcations may not happen [108]. For this reason alone, the route to chaos mechanism as coupling strength is increased for a single laser with feedback [26] can not proceed in the same way. The left column of Fig. 18 shows the calculated Floquet multipliers of the symmetric undamped relaxation oscillations at different bifurcations.

Examining matrix (3.5) reveals more. As $T$ is a large parameter it is tempting to consider the bottom row of matrix (3.5) as small, therefore the right eigenvector is purely in the $N_D$ direction hence the limit cycle is born in the $(\phi_D, A_D)$ plane. For small $\epsilon$, adding an oscillation to the electric fields such as $A_D = \epsilon \cos(\omega_H t) + \mathcal{O}(\epsilon^2)$, $\phi_D = \pi + \epsilon (\omega/A_s) \sin(\omega_H t) + \mathcal{O}(\epsilon^2)$ where $\phi_S = 2 \omega_A t$ yields the following:

$$E(t) = \left(\frac{1}{2}\right) \left[ A_S + \epsilon \cos(\omega_H t) \right] e^{i\omega_A t} e^{i\left(\frac{\pi}{2}\right)} e^{i\left((-\frac{1}{2})\sin(\omega_H t)\right)}$$

$$= \left(\frac{i}{2}\right) \left[ A_S + \epsilon \cos(\omega_H t) \right] e^{i\omega_A t} \left[ 1 + i \frac{\epsilon}{A_S} \sin(\omega_H t) \right]$$

$$= \left(\frac{i}{2}\right) \left[ A_S e^{i\omega_A t} + \epsilon e^{i\omega_A t} \right] + \mathcal{O}(\epsilon^2) \quad (3.8)$$

where $\omega_D = \omega_A + \omega_H$. Therefore as seen, a limit cycle born into $(\phi_D, A_D)$ plane results in an electric field with two optical colours. For high coupling ($\kappa > GH1$) evident in Eq. (3.6) and plotted in Fig. 19, the frequency of oscillation scales with twice the coupling strength. This is the mode beating between in-phase and anti-phase one colour states as discussed in Sec. 3.2.

For weaker coupling strengths, the situation is different. As $\kappa$ is small of comparable size to $(1/T)$, it cannot be argued away from Eq. (3.5) as easily. The non-degenerate left eigenvalue is now in the $\phi_D$ direction which means that the limit cycle is born into the $(A_D, N_D)$ plane. As above, the effect on the electric field is examined by adding a small oscillation in this plane.

$$E(t) = \left(\frac{1}{2}\right) \left[ A_S + \epsilon \cos(\omega_H t) + \mathcal{O}(\epsilon^2) \right] \exp[i\omega_A t]$$

$$= \left(\frac{1}{2}\right) \left[ \epsilon \exp[i\omega_H t] + A_S \exp[i\omega_A t] + \epsilon \exp[i\omega_D t] \right] \quad (3.9)$$

where $\omega_D = \omega_A - \omega_H$ and $\omega_D = \omega_A + \omega_H$. A limit cycle born into the $(A_D, N_D)$ plane results in three optical colours. As evident in Eq. (3.6) for small $\kappa$, the colours are distanced by the relaxation oscillations of a free-running laser. This analysis is graphically illustrated in Fig. 20.

Above, we showed that at low coupling strength and $0 < C_p < \pi$, the anti-phase state looses transversal stability resulting in an oscillatory solution of three or more odd number of optical
colours centred at the anti-phase frequency and distanced by the relaxation oscillations. Physically the damping for the equilibria of positive pushed inversions become weaker and weaker as the coupling is increased until the oscillations are not damped at all. These are the symmetric undamped relaxation oscillations states. An integration of a typical stable undamped relaxation oscillation for weak coupling is displayed in Fig. 21. As can be seen, the inversions are centred slightly above zero and oscillates with the frequency of the relaxation oscillations. Shifting the output of laser one by half the period of oscillation gives an identical state to laser two and vice versa. For this reason, they are considered symmetric and satisfy symmetry (3.7).

Ref. [79] lists the types of non-isolated codimension one bifurcations an S-cycle can undergo for even symmetry group. All these are observed to occur and will feature in the discussion of Sec. 4.2. One such bifurcation is when the Floquet multiplier crosses the unit circle at 1. This results in either a saddle-node of limit cycles or pitchfork of limit cycle. When a saddle-node of limit cycles occurs the S-type cycle disappears. When a pitchfork of limit cycle occurs the symmetry (3.7) is broken resulting in a related cycle. A typical stable example of which is shown in Fig. 22. In Ref. [81] these are called M-type cycles as any event that occurs to one is simultaneous mirrored in the conjugate cycle. We refer to these periodic orbits as symmetry-broken undamped relaxation oscillations. Here the two lasers are still locked to the same set of optical colours distanced by the relaxation oscillations but can have significant different intensity at each frequency. Another type of bifurcation that can occur to S-type cycles are when two complex conjugated Floquet multipliers cross the unit circle in the positive direction giving birth of a symmetric 2-torus [79].

### 3.4 QuasiPeriodic

Quasi-periodic orbits on a torus are well-known to occur for mutually coupled lasers [61]. At low coupling strengths a special type of this behaviour is present. The relaxation oscillation frequency still dominates but a new non-commensurate frequency is introduced.

A symmetric 2-torus is born after two Floquet multipliers cross the unit circle for an S-type cycle. A typical example of these dynamics is shown in Fig. 23. Here we see that both lasers are locked to the same optical frequencies with equal intensity in each laser for each frequency. In addition the timetrace for both the carrier densities and intensities of each laser are identical with a time shift, Eq. (3.7). Fig. 24 shows a symmetry-broken variant of quasi-periodic behaviour observed numerically at weak coupling strengths. The relaxation oscillation frequency is still clearly present. For these parameters, every 5 ns or so, seemingly constant intensity dynamics explodes into oscillations. The inversions for in-phase and anti-phase symmetric one-colour is obtained from Eq. (2.8) and plotted as the black dashed lines in the inversion plot of Fig. 24.

The lengths of time from the centre of one explosion to the next are manually obtained by looking at several similar timetraces as Fig. 24 with slightly different coupling phases. This is shown in Fig. 25. Scalings\(^1\) are added by fixing a curve to highest period. These will have implication on conclusions in Sec. 4.2, however for now we only point out that the second commensurate frequency needed from quasi-periodic behaviour comes from infinite period.

\(^1\)Table 7.4.1 in [107]
Figure 14: Diagram shows in-phase one-colour states (red), anti-phase one-colour states (blue), symmetry-broken one-colour states (purple) and symmetric two-colour states (green) at $\kappa = 0.1$ for $2\pi$ range of $C_p$. Labels H, P and PLC are Andronov-Hopf, pitchfork and pitchfork of limit cycles bifurcations respectively.
Figure 15: Time-traces (top panels) and frequency spectra (bottom panels) showing symmetric (left column) and symmetry-broken (middle and right columns) CLMs for $\tau = 0.1$, $\kappa = 0.3$ and $C_p = 0.33\pi$. These parameters are consistent with a point in region 7 of Fig. 27.
Figure 16: Numerical solution of system (2.3) for $\tau = 0.3, \kappa = 0.15, C_p = 0.33\pi$ which is consistent with region 4 of Fig. 27. The top left panel contains the magnitude of the electric fields, and the top right their inversions. The bottom panel shows the optical spectrum relative to the frequency of the free running lasers. The dot-dashed vertical lines are the frequencies calculated from ansatz (3.3).

Figure 17: Transition between in-phase (left) and anti-phase (right) CLMs for $\tau = 0.2, \kappa = 0.4$. The in-phase ($\omega_{in} = -74.19 \text{ GHz}$) and anti-phase ($\omega_{an} = +49.46 \text{ GHz}$) one-colour frequencies are obtained from Eq. (3.2).
Figure 18: Six diagrams showing the critical Floquet multipliers (red) and non-critical (green) calculated using AUTO [22] for $\tau = 0$ at several bifurcation points. Left column are for S-type relaxation oscillation cycles and right column (d-f) are for M-type relaxation oscillation cycles. $(\kappa, C_p) =$ (a) $\{0.000010192, 0.4 \pi\}$, (b) $\{0.00160342, 0.65 \pi\}$, (c) $\{0.0016, 0.7344 \pi\}$, (d) $\{0.002, 0.60009 \pi\}$, (e) $\{0.002, 0.14543 \pi\}$, (f) $\{0.00332465, 0.4 \pi\}$. 
Figure 19: Large left diagram (a) demonstrates the frequency of oscillation scales with twice the coupling strength $\kappa$ for moderate coupling strength with the dashed line being subcritical. Top-right (b) shows the plane in which the limit cycle is created and bottom-left (c) exhibits the resultant optical state.

Figure 20: Large left diagram (a) demonstrates the frequency of oscillation is the relaxation oscillations of a free-running laser at low coupling strength with dashed lines being subcritical. Top-right (b) shows the plane in which the limit cycle is created and bottom-left (c) exhibits the resultant optical state.
Figure 21: Timetraces showing the dynamics for $\tau = 20$, $\kappa = 0.0009$, $C_p = 0.4\pi$. (symmetric undamped relaxation oscillations)
Figure 22: Timetraces showing the dynamics for $\tau = 20$, $\kappa = 0.003$, $C_p = 0.4\pi$. (symmetry-broken undamped relaxation oscillations)
Figure 23: Timetraces showing the dynamics for $\tau = 20$, $\kappa = 0.002$, $C_p = 0.7\pi$. (symmetric quasi-periodic dynamics)
Figure 24: Timetraces showing the dynamics for $\tau = 20$, $\kappa = 0.002$, $C_p = 0.16\pi$. (symmetry-broken quasi-periodic dynamics)
Figure 25: Scaling laws for homoclinic and infinite period bifurcations at $\tau = 0.0$, $\kappa = 0.002$ and $C_p = 0.1227\pi$. Square boxes were visual observations from numerics of the largest period oscillation.
Chapter Appendix

3.A Symmetric and Symmetry Broken

CLMs or one-colour states are defined in the main text by the ansatz (3.1) for the delayed coupled system of ODEs (2.3). The following equivalence relation characterise symmetric CLMs whilst symmetric-broken CLMs are the complement class.

**Theorem 3.A.1** The following three conditions are equivalent.

(i) \( N_1 = N_2 \)

(ii) \( A_1 = A_2 \)

(iii) \( \delta_A = 0, \pi \)

**Proof** Considering equilibrium solutions for \( N_{1,2} \) for Eqs. (2.3c) (2.3d) yields

\[
A_1^2 = \frac{P - N_1}{2N_1 + 1} \quad A_2^2 = \frac{P - N_2}{2N_2 + 1}
\]

(3.10)

Since \( A_{1,2} \geq 0 \), functions (3.10) give 1-1 relation between \( N_{1,2} \) and \( A_{1,2} \) therefore assuming condition (i) implies (ii).

Noting functions (3.10) are involutions, therefore \( N_{1,2} \) can be switched with \( A_{1,2}^2 \). Then using the same logic as above, assuming (ii) implies (i). Substituting ansatz (3.1) into the equations for the electric fields (2.3a) (2.3b) gives two complex conditions

\[
\frac{A_1}{A_2} (i\omega_A - (1 + i\alpha)N_1) = ke^{-iC_P e^{i\omega_A \tau}} e^{+i\delta_A} \\
\frac{A_2}{A_1} (i\omega_A - (1 + i\alpha)N_2) = ke^{-iC_P e^{i\omega_A \tau}} e^{-i\delta_A}
\]

(3.11)

(3.12)

Assuming (ii) then the ratio of the two amplitudes disappear in Eqs. (3.11)(3.12). As (ii) implies (i), both left hand sides are equal which gives the condition

\[ e^{+i\delta_A} = e^{-i\delta_A} \]

Therefore \( \sin \delta_A = 0 \) and \( \delta_A = 0, \pi \). Therefore assuming (ii) implies (iii).

Finally, assuming (iii), i.e \( \delta_A = 0, \pi \), the right hand sides of Eqs. (3.11)(3.12) are equal. Combining these two equations and splitting real and imaginary parts gives the following two real conditions:

\[
\frac{A_1}{A_2} = \frac{N_2 A_2}{N_1 A_1} \\
(\omega_A - \alpha N_1) \frac{A_1}{A_2} = (\omega_A - \alpha N_2) \frac{A_2}{A_1}
\]

Substituting one into the other, after cancellations implies \( N_1 = N_2 \) or condition (ii). This suffices as a proof.

38
The above conditions justify the distinction between symmetric and symmetry-broken states used in the thesis. Furthermore it distinguishes between “in-phase” and “anti-phase” one-colour symmetric states.

3.B The frequency $\omega_A$ for one-colour states

In order to obtain the frequency $\omega_A$ appearing in the one-colour ansatz (3.1), we first derive an explicit relation between the inversion $N_1^c$ and $N_2^c$ as a function of $\omega_A$. Inserting (3.1) into the equations for $\dot{E}_1$ (2.3a) and $\dot{E}_2$ (2.3b) yields the matrix equation

\[
\begin{pmatrix}
\alpha N_1^c - i \omega_A \\
ke^{-i e \delta A} & \alpha N_2^c - i \omega_A
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 e^{i \delta A}
\end{pmatrix}
= 0
\] (3.13)

where we have used the notation $\alpha = 1 + i \alpha$. This equation has a non-trivial solutions if its determinant vanishes. For $\omega_A$ this leads to the complex condition

\[
\alpha N_1^c N_2^c - i \omega_A (N_1^c + N_2^c) = \frac{[ke^{-i e \delta A}]^2 + (\omega_A)^2}{\alpha}
\] (3.14)

This allows us to explicitly express $N_1^c$ and $N_2^c$ as functions of $\omega_A$ via

\[
N_{1/2}^c = -\frac{\text{Im}[A^* Q_A]}{2 \omega_A} \pm \sqrt{\left(\frac{\text{Im}[A^* Q_A]}{2 \omega_A}\right)^2 - \text{Re}[Q_A]}
\] (3.15)

with

\[
Q_A = \left[ke^{-i e \delta A}\right]^2 + (\omega_A)^2.
\] (3.16)

Note that the sign $\pm$ in (3.15) was chosen in such a way that $N_1^c \geq N_2^c$, however the alternative choice with $\mp$ is equally possible and would imply $N_1^c \leq N_2^c$.

We can now express $A_{2/1}^c$ in two different ways via,

\[
\frac{(P - N_1^c)(2N_1^c + 1)}{(2N_2^c + 1)(P - N_2^c)} A_2^c = \frac{A_2^c}{A_1^c} = \frac{\omega_A - \alpha N_1^c}{\kappa^2}.
\] (3.17)

where we have used the two Eqs. (3.10) on the left hand side, and the magnitude of the complex expression Eq. (3.11) on the right hand side. We then substitute the explicit functions (3.15) of $N_{1/2}^c$ into Eq. (3.17) above which gives an implicit solution for the frequency $\omega_A$ of the one-colour state for the system with delay time $\tau$. The frequency $\omega_A$ is finally obtained numerically from this transcendental expression with a secant method yielding the dot-dashed vertical lines in Fig. 15.

3.C The frequencies $\omega_A$ and $\omega_B$

We show, how the frequencies $\omega_A$ and $\omega_B$ appearing in the ansatz (3.3) for the symmetry-broken two-colour states can be obtained analytically. Inserting (3.3) into the equations for $\dot{E}_1$ and $\dot{E}_2$ and sorting the terms for the two frequencies $\omega_A$ and $\omega_B$ yields in addition to Eq. (3.13), a second matrix equation

\[
\begin{pmatrix}
\alpha N_1^c - i \omega_B \\
ke^{-i e \delta B} & \alpha N_2^c - i \omega_B
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2 e^{i \delta B}
\end{pmatrix}
= 0
\] (3.18)

Eqs. (3.13) and (3.18) are identical, except for the fact that $\omega_A$ is replaced by $\omega_B$. The same arguments used in App. 3.B for deriving the expressions for $N_{1/2}^c(\omega_A)$ in Eq. (3.15) also hold
for $N_{1/2}^c(\omega_B)$. Therefore explicit expressions for $N_{1/2}^c(\omega_B)$ are obtained by replacing $\omega_A$ by $\omega_B$ in Eq. (3.15). We also have $N_1^c(\omega_A) = N_1^c(\omega_B)$ and $N_2^c(\omega_A) = N_2^c(\omega_B)$. Therefore we can parametrically plot $(N_1^c(\omega), N_2^c(\omega))$ as a function of $\omega$ in the $(N_1, N_2)$ plane as shown in the blue curve of Fig. 26 and the self intersection point of this curve determines the inversions $N_1^c$ and $N_2^c$ and the frequencies $\omega_A$ and $\omega_B$ of the symmetry-broken two-colour states. We stress that we did not make use of the equations for $N_1$ and $N_2$ of system (2.3), and because the ansatz (3.3) is only approximately true, the obtained frequencies are not exact.

3.3 Anti-Phase Linearization

Using the instantaneous limit $\tau = 0$, and writing the system (2.3) as a sum “S” and difference “D” as in main text (2.13) a six dimensional system is obtained.

\[
\begin{align*}
\dot{\phi}_D &= \alpha N_D + \frac{2\kappa}{A_S - A_D^2} \\
&\quad \times \left(2A_D A_S \sin(C_P) \cos(\phi_D) - (A_D^2 + A_S^2) \sin(\phi_D) \cos(C_P) \right) \quad (3.19a) \\
\dot{A}_D &= \frac{N_D A_S + N_S A_D}{2} - \kappa A_D \cos(C_P) \cos(\phi_D) - \kappa A_S \sin(C_P) \sin(\phi_D) \quad (3.19b) \\
\dot{A}_S &= \frac{N_D A_D + N_S A_S}{2} + \kappa A_S \cos(C_P) \cos(\phi_D) + \kappa A_D \sin(C_P) \sin(\phi_D) \quad (3.19c) \\
T\dot{N}_D &= -N_D - A_D A_S (1 + N_S) - N_D (A_D^2 + A_S^2) \\
T\dot{N}_S &= 2P - N_S - \frac{(A_D^2 + A_S^2)}{2} - N_D A_D A_S - N_D (A_D^2 + A_S^2) \quad (3.19e)
\end{align*}
\]

$\phi_S$ decouples from the system reducing the order. Linearization is done around the anti-phase state using the lower first term of Eq. (2.8), Eq. (2.5) and taking $\phi_D = \pi$ one obtains the following Jacobian matrix

\[
M := \begin{bmatrix}
2\kappa \cos(C_P) & -2\frac{\kappa}{\pi} \sin(C_P) & 0 & \alpha & 0 \\
-2\frac{\kappa}{\pi} \sin(C_P) & 2\kappa \cos(C_P) & 0 & \frac{\pi}{\gamma} & 0 \\
0 & 0 & 0 & 0 & \beta \\
0 & -\frac{4}{\gamma} & 0 & -\frac{2P+1}{T}\beta & \frac{\pi}{\gamma} \\
0 & 0 & -\frac{4}{\gamma} & 0 & -\frac{2P+1}{T}\beta
\end{bmatrix} \quad (3.20)
\]

where for neatness $\beta = (1 + 2\kappa \cos(C_P))^{1/2}$ and $\gamma = (P - \kappa \cos(C_P))^{1/2}$. We seek to determine the eigenvalues by writing the characteristic equation $\det(M - \lambda I) = 0$. Taking advantage of some zero terms in matrix $M$, one can extract the following conjugate eigenvalues.

\[
\begin{align*}
\lambda_1 &= -\left(\frac{2P + 1}{2T\beta^2}\right) + \left(\frac{2P + 1}{2T\beta^2}\right)^2 - \frac{\gamma^2}{\pi} \quad (3.21) \\
\lambda_2 &= -\left(\frac{2P + 1}{2T\beta^2}\right) - \left(\frac{2P + 1}{2T\beta^2}\right)^2 - \frac{\gamma^2}{\pi} \quad (3.22)
\end{align*}
\]

These are not critical $\text{Re}(\lambda) \neq 0$ in our study. The imaginary term are the relaxation oscillations (2.12) which are degenerate with other conjugate pair of Eq. (3.20) at $\kappa = 0$. Removal of
eigenvalues (3.21) reduces the order of the linearization matrix.

\[
\begin{pmatrix}
2\kappa \cos(C_p) & -2\frac{\beta}{\beta} \kappa \sin(C_p) & \alpha \\
2\frac{\gamma}{\alpha} \kappa \sin(C_p) & 2\kappa \cos(C_p) & \frac{\gamma}{\alpha} \\
0 & -\frac{1}{2} \gamma \beta & -\frac{1}{2} \frac{2P+1}{\beta} \beta
\end{pmatrix}
\]  

(3.23)

which can be written as the following cubic polynomial.

\[
a\lambda^3 + b\lambda^2 + c\lambda + d = 0
\]

(3.24)

where

\[
\begin{align*}
a &= T\beta^2 \\
b &= (2P + 1) + 4\kappa T\beta^2 \cos(C_p) \\
c &= 4\kappa^2 T\beta^2 - 2(2P + 1)\kappa \cos(C_p) + 2\gamma^2 \beta^2 \\
d &= 4\kappa^2 (2P + 1) + 4\kappa \gamma^2 \beta^2 (\alpha \sin(C_p) - \cos(C_p))
\end{align*}
\]

Unfortunately cubic Eq (3.24) is not readily solved. Adopting a Routh-Hurwitz technique to find Hopf bifurcation \( \lambda = 0 + i\omega \), the condition \( ad = bc \) is obtained which is used to provide parameter values for \( \kappa, C_p \) in Figs. 19 20. The frequency in these figures and Eq. (3.6) is obtained from \( \omega = \sqrt{c/a} \).
Figure 26: Parametric plot of \((N_1^c(\omega_A), N_2^c(\omega_A))\) (blue line) according to Eq. (3.15) using the parameter values of Fig. 16. The frequencies \(\omega_A\) and \(\omega_B\) and the inversions \(N_1^c\) and \(N_2^c\) of the symmetry-broken two-colour states are obtained from the point where the blue line self-intersects. The green line shows a parametric plot of \((N_2^c(\omega_A), N_1^c(\omega_A))\).
4 Bifurcation Picture

4.1 High Coupling

In this section the bifurcation structure at $\tau = 0$ is explored using the 5 dimensional model (2.16). In the reduced coordinate system (2.14), one-colour states become fixed point solutions and the two-colour states become limit cycles. This reduction in complexity allows us use continuation software AUTO [22] to obtain a comprehensive overview of the involved bifurcations.

In Fig. 27, the bifurcation diagram with the two bifurcation parameters of coupling phase $C_p$ and coupling strength $\kappa$ is presented. Due to the parameter symmetry

$$(q_x, q_y, C_p) \rightarrow (-q_x, -q_y, C_p + \pi)$$

it is sufficient to consider the parameter range of $C_p$ in the interval $[0, 1\pi)$ only. The (codimension one) bifurcation lines in the diagram separate the parameter space into eight distinct regions. Only bifurcations which affect stable dynamical states are plotted. In region 1 and 2 in-phase one-colour states and anti-phase one-colour states are stable respectively. For the new coordinates (2.14), symmetric one colour states are confined to the $q_x$ axis ($q_y = 0; q_z = 0$), with $q_x > 0$ in the in-phase case as shown in Fig. 14.

At very high coupling strengths ($\kappa > PH1$), the in-phase region 1 and the anti-phase region 2 are separated by two supercritical Hopf bifurcations in close proximity as shown in the upper left panel of Fig. 28. Between these two Hopf bifurcations a stable limit cycle of low amplitude and high frequency exists which corresponds to the symmetric two-colour dynamics seen in Fig. 17 of Sect. 3.2, where the torus bifurcations (cf. Fig. 35) of the original system (2.3) have become the Hopf bifurcations in the reduced system. In Fig. 27 these two vertical Hopf bifurcation lines appear as a single line at the scale of the diagram with the tiny region 3 of symmetric two-colour states nested between them. Fig. 14 shows that the two symmetric one-colour states are connected via symmetry-broken one-colour states [28] and symmetric two-colour states for moderate coupling strengths. The near vertical green lines highlights that symmetric two colour states occur over a very small region of the coupling phase $C_p$. In addition symmetry-broken one-colour states ($q_y \neq 0; q_z \neq 0$) are stable in region 6 which is accessible from the symmetric one-colour states via a supercritical bifurcation at the top of the blue pitchfork line between the SNP and PH1 points.

A sketch of the situation in the vicinity of the PH1 point is provided in the left part of Fig. 30, where one of these supercritical Hopf bifurcations and the supercritical pitchfork bifurcation intersect. The second supercritical Hopf bifurcation although nearby does not play a role. The PH1 point is a pitchfork Hopf codimension two bifurcation and analytical expressions are obtained in App. 4.A. In order to classify this bifurcation, we note that the pitchfork Hopf and the Hopf-Hopf bifurcation share the same reduced normal form. PH1 corresponds to a Hopf-Hopf "simple" case III of [63] or equivalently case II/III of [37]. Figure 30 illustrates that this pitchfork Hopf point is responsible for the symmetry-broken two colour dynamics which originates between a pitchfork of limit cycles (PLC) and a supercritical Hopf bifurcation ($2H_{sb}$). Symmetry-broken two-colour states are stable in region 4 of Fig. 27. The point labelled SNP is a saddle-node pitchfork codimension two bifurcation. It shares the same reduced normal form as a generalised (Bautin) Hopf bifurcation.
without rotation. A schematic is provided in Fig. 31. At this point the supercritical pitchfork bifurcation becomes subcritical, thus spawning the tri-stable region 7, where both symmetry-broken and symmetric one-colour states are stable. An integration with noise of the original system (2.3) showing the three stable one-colour states was presented in Fig. 15 of Sec. 3.1. The third corner of symmetry-broken one-colour states is closed by a saddle-node Hopf (also known as fold Hopf) codimension two bifurcation labelled SNH in Fig. 27. At this point a saddle-node and a Hopf bifurcation meet tangentially and also a torus bifurcation line emerges from this point. The unfolding of saddle-node Hopf bifurcations have been studied extensively in the literature, and we identify the SNH point with the third case in [63]1. A topological bifurcation diagram is supplied in Fig. 32. These three codimension two bifurcations organise the multi-stabilities.

Next we provide a series of cuts of constant coupling strength $\kappa$ across the diagram (Fig.27) to further explain the dynamics in each of the regions. In Fig. 29, a plot at $\kappa = 0.1$ of the inversions $N_{1,2}$ across the entire $1\pi$ span of the coupling phase $C_p$ is presented. This plot illustrates that symmetry-broken two-colour states are stable over a large range of $C_p$. The bottom inset highlights region 5 which is relatively small at $\kappa = 0.1$. In this region a symmetric one-colour is stable in addition to the symmetry-broken two-colour states. In the right-most inset a blow-up of region 3 is provided. The limit cycle born after a supercritical Hopf bifurcation is still invariant under $Z_2$ symmetry (2.15) and therefore obeys

$$\mathbf{q}(t) = \mathbf{q}(t + \frac{\tau}{2}),$$

where $\tau$ is the period of the limit cycle. In a projection to the $(q_x, q_y, q_z)$ coordinates, the limit cycles therefore form rings around the $q_z$ axis, as shown in the lower right panel of Fig. 28. These limit cycles corresponds to the symmetric two-colour states. They lose stability at a pitchfork of limit cycles bifurcation and create the large regions 4 and 5 of symmetry-broken two-colour states. In the bifurcation diagram (Fig. 27) the GH point is a codimension two generalised Hopf (Bautin) bifurcation. As shown in bottom right panel of Fig. 28, the scenario remains very similar for $\kappa$ below this point. However here the supercritical Hopf bifurcation becomes subcritical and a saddle-node of limit cycles bifurcation occurs. Finally in the top right panel of Fig. 28, we look at an inversion cut of much higher $\kappa$ just beneath the SNP point. Here the supercritical Hopf and supercritical pitchfork of limit cycles bifurcations occur in close proximity at the point B. In contrast to Fig. 29, symmetry-broken one-colour states are stable limited on the left by a saddle-node bifurcation and on the right by a supercritical Hopf bifurcation. Region 7 of Fig. 27 corresponds the small area of hysteresis between these symmetry-broken one-colour and symmetric one-colour states.

Fig. 27 is a principal result of this research. The appendix Fig. 37 contains the same parameter map considering the coupling as a complex parameter and Fig. 38 (top) includes bifurcations that act on unstable states. The remaining two codimension two points plotted in Fig. 27 are a period-doubling torus (PDT), and a second pitchfork Hopf (PH2) bifurcation equivalent to the Hopf-Hopf “difficult” case VI of [63] which is the same as case VIa of [37]. This completes the bifurcation picture for moderate to high coupling strength. We note that different dynamics occurs for lower coupling strengths which is the topic of the next section.

1[63] does not enumerate the cases. Here we refer to the case of Figure 8.16 in the 3rd version of the book.
4.2 Low Coupling

In this section a full bifurcation picture Fig. 33 for weak coupling strength at $\tau = 0$ is presented. The reduced coordinate system of Sec. 2.3 which importantly contains no singularities made it possible to use continuation software AUTO [22] to detect the exact bifurcations positions by small changes of one parameter and subsequently following them by varying two. Continuation methods are well known to the community and will not be summarised here, for an introduction see [21].

In Fig. 33 the green codimension one bifurcation that acts on anti-phase symmetric states is super-critical from PH2 to GH2 and afterwards subcritical increasing to coupling strengths outside the low coupling study (cf. Fig. 19 and Sec. 4.1). This is the Hopf bifurcation that leads to symmetric undamped relaxation oscillations, (cf. Fig. 21). Fig. 20 (a) showed that the frequency of this line versus the coupling phase is completely flat at the relaxation oscillations of the lasers. In Fig. 33 with coupling strength as ordinate, it bows slightly. The red line slightly above and running alongside is a pitchfork of limit cycles bifurcation introduced in Sec. 3.3. This breaks the symmetry (3.7) creating a relatively large region of stable symmetry-broken undamped relaxation oscillations (cf. Fig. 22). To the right the pitchfork of limit cycles gives way to a symmetric torus bifurcation mentioned in Sec. 3.4 and represented by the sloped orange line. This bifurcation also acts on the symmetric undamped relaxation oscillations and creates a sizeable region of symmetric 2-tori (cf. Fig. 23). Its noted that unlike other categorised areas of diagram Fig. 33 this region is not as homogeneous and contains other wispy structures with features including symmetry-broken 2-torus and symmetric 3-tori. Study of these would require a higher order investigation with an ability to continue quasi-periodic behaviour. This region is however predominately symmetric 2-tori both internally and at its bottom, top, and right boundaries. The right-hand side of both the regions of symmetric undamped relaxations and symmetric tori contains a sizeable area of bi-stability with a symmetric one colour state. This contrast with the case of higher coupling, Sec. 4.1 where multi-stabilities occurred between symmetric and symmetry-broken states. The symmetric undamped relaxation oscillations are bounded on the right by the purple saddle-node of limit cycles bifurcation mentioned in Sec. 3.3. This emanates from GH2 which is a generalised Hopf or Bautin codimension two bifurcation. The right boundary, dashed brown line, of symmetric quasi-periodic behaviour was determined using numerical sweeps. A strong note of caution should therefore be added to this boundary. The tori at this line terminates in an undetermined fashion. No such scaling as in Fig. 25 was observed by us. The region of symmetry-broken undamped relaxation oscillations is bounded on the left by a torus bifurcation which creates a region of symmetry-broken tori of the form in Fig. 24. To see how the symmetry-broken tori region is bounded on the left we need to look closer at the codimension two points.

PH2 is a pitchfork Hopf codimension two bifurcation, it shares the same normal form as a Hopf-Hopf difficult case VI of [64] which is the same as case VIa of [39]. A blow-up sketch of this codimension two bifurcation and its vicinity is provided in Fig. 34. Co-dimension one bifurcation that act on unstable states appears as dashed lines. Firstly it is observed that the apparent single lower pitchfork bifurcation in Fig. 33 is in fact two subcritical pitchfork bifurcations in close proximity. The rightmost pitchfork bifurcation which in the case of $0 < C_p < \tau$ acts on in-phase one-colour states complicates the picture but does not play an active role in the PH2 point story. This is the blue vertical line in Fig. 33 which extends from PH2 to higher coupling and forms the right boundary of symmetric one-colour states. As seen in Fig. 34 and Fig. 33, it limits several regions of multi-stability between in-phase one-colour states and anti-phase one-colour states [128], symmetric undamped relaxation oscillations, symmetry-broken undamped relaxation oscillations, and symmetry-broken tori. These multi-stabilities are so small that they are invisible at the scale of Fig. 33. The left pitchfork bifurcation in Fig. 34 is subcritical and acts on stable anti-phase one...
colour states for $0 < C_p < \pi$ from degenerate point $C_p^*$ to PH2. Above PH2 point it only acts on unstable anti-phase states and therefore is represented by a dashed line. Skirting this line is an infinite period bifurcation which limits the region of symmetry-broken tori, Fig. 24 on the left side. This bifurcation was determined by observing the scalings at which the highest frequency of the torus approaches infinity. In Fig. 25 the highest frequency was manually obtained from timetraces for several different coupling phases at distances from a point on the pitchfork/infinite period line. It is shown that it scales at a rate consistent with an infinite period bifurcation rather than an homoclinic of tori [107] which is expected in the generic pitchfork-Hopf of this type. The green horizontal Hopf line in Fig. 34 which creates the symmetric undamped relaxation oscillations (cf. Fig. 21) acts on stable anti-phase states (cf. Fig. 8) to the right of PH2 and unstable anti-phase states to the left. A green Hopf line which acts on unstable symmetry-broken one-colour states extends vertically. Both the pitchfork of limit cycles which creates the symmetry-broken undamped relaxation oscillations (cf. Fig. 22) and the torus bifurcation which creates symmetry-broken tori (cf. Fig. 24) both emanate from the pitchfork-Hopf bifurcation. For this reason the PH2 point has a dominating effect on the dynamics at low coupling strengths. The other labelled codimension two bifurcations in Fig. 33 are a generalised Hopf bifurcation (cf. Figs. 19 and 20) and period doubling torus points (PDT). Fig. 36 schematically illustrates the dynamics surrounding PDT points.

The Lyapunov exponent is a measure of the rate of separation of infinitesimally close trajectories as a dynamical system evolves and therefore gives a measure of the sensitivity to minute changes in the initial conditions. A maximal Lyapunov exponent greater than zero is seen as an indication of chaos. In this study, for several thousand points spanning the parameter map Fig. 33, the maximal Lyapunov exponent is estimated using a method adapted from Ref [106]. A contour plot is then made and overlaid on top of the bifurcation diagram. Areas shaded in grey in Fig. 33 have positive Lyapunov exponent. Towards the top of Fig. 33, symmetry-broken undamped relaxation oscillations undergo several period doublings before becoming chaotic. Both the symmetric and symmetric-broken tori disintegrate into chaotic dynamics as the coupling strength increases. The period doubling bifurcations that act on symmetry-broken undamped relaxation oscillations and the right and left torus bifurcation boundaries interact tangentially at several co-dimension two period doubling torus bifurcations. A explanatory sketch of a generic period doubling torus is provided in Fig. 36. This completes the bifurcation picture for low coupling strength.
Figure 27: Bifurcation diagram of system (2.16) in the \((C_p, \kappa)\) parameter plane. The bifurcation lines separate parameter space into eight distinct dynamical regions. The codimension one bifurcation lines are organised by a set of codimension two points. The labelling used for the dynamical regions and for the codimension two points are explained in the left and right legends below the main graph. Only bifurcations which effect stable states are drawn, Figs. 38 (top) includes bifurcations acting on unstable states.
Figure 28: Several cuts of constant $\kappa$ as indicated across Fig. 27 are shown. The top panels and lower-left panel show bifurcation diagrams of $N_{1,2}$ versus $C_p$. Solid lines indicate stable states and dashed lines unstable states. Symmetric in-phase and anti-phase one-colour states are red and blue respectively. Symmetry-broken one-colour states appear as purple lines whilst symmetry-broken limit cycles are orange. Green lines indicate symmetric limit cycles. Hopf, pitchfork, pitchfork of limit cycles, saddle-node, saddle-node of limit cycles are denoted by H, P, PLC, SN, SNLC. Point B contains several bifurcations, see main text. The lower-left panel shows a 3-dimensional graph in $q$-vector space (2.14) showing symmetric limit cycles (green) between in-phase (red) and anti-phase (blue) one-colour states. The orange ring marks the pitchfork of limit cycles. The $C_p$ range corresponds to the same as right-most inset of Fig. 29. Symmetry-broken limit cycles not drawn.
Figure 29: An inversion cut at $\kappa = 0.1$ across the entire $1\pi$ range of $C_p$. Colours and labels are as in Fig. 28. T represents a torus bifurcation. Insets are blow-ups of the regions indicated.
Figure 30: Sketch of bifurcation structure in vicinity of the pitchfork-Hopf codimension two point, labelled PH1 in Fig. 27. Subscripts $in$, $an$ and $sb$ refer to bifurcations acting on in-phase, anti-phase and symmetry-broken states respectively. The solid bifurcation lines affect stable states, while the dashed lines only affect unstable states.
Figure 31: Schematic showing a subcritical pitchfork bifurcation changing to a supercritical pitchfork. The saddle-node pitchfork codimension two bifurcation shares the same reduced normal form as a generalised (Bautin) Hopf bifurcation without rotation.

Figure 32: Topological bifurcation of the dynamical states in the vicinity of the SNH point. This is equivalent to Fig. 8.16 of [63]. The position of heteroclinic connection was not determined in this work.
Figure 33: Parameter map for two weakly coupled lasers. Codimension one bifurcation, labelled below-right, appear as lines which separate the parameter space into regions of different dynamical behaviour. The numbers inside the circles link to states in Figs. 7, 8, 21, 22, 23, 24, 48, and 49. Co-dimension two bifurcation appear as labelled points whose full names are below-left of the main graph. Areas shaded in grey have positive Lyapunov exponent.
Infinite Period

$P_{an}$

$H_{sb}$

$T_{sb}$

$P_{in}$

$PLC$

$H_{an}$

$\kappa = 0$

$C_p$

$\kappa$

Figure 34: Sketch of the dynamical picture in the vicinity of the PH2 codimension two bifurcation point. Dashed lines are codimension one bifurcations that do no act on stable dynamical states. Two numbers inside a circle represents an area of bi-stability.
Figure 35: A supercritical torus bifurcation acts on a stable limit cycle creating a stable torus and unstable cycle. More than 3 dimensions allows the two to decompose.

Figure 36: A torus bifurcation acts on a limit cycle adding a characteristic frequency. As two parameters are varied a special point exists where the new frequency is either double or half the characteristic frequency leading to a period doubling.
Chapter Appendix

4.A pitchfork-Hopf codimension two points

The codimension two point PH1 in Figs. 27 and 30 plays an important role in organising the bifurcation structure of system (2.16). Pitchfork and symmetric Hopf bifurcation lines are obtained in [128], here we derive analytical expressions for their intersection and thus the location of the two pitchfork Hopf points, PH1 and PH2 in the \((C_p, \kappa)\) parameter plane.

Let us first evaluate the Jacobian of (2.16) at an in-phase one-colour state with \(q_x = (2P + \kappa \cos(C_p))/(1 - 2\kappa \cos(C_p)) > 0, \ q_y = q_y = 0\) and \(N_1 = N_2 = -\kappa \cos C_p\). In the slightly transformed coordinates \((q_x, N_S, q_y, q_z, N_D)\), where \(N_S = (N_1 + N_2)/2\) and \(N_D = (N_2 - N_1)/2\), the Jacobian \(J\) has then the form

\[
J = \begin{pmatrix}
J_1 & 0 \\
0 & J_2
\end{pmatrix},
\]

(4.3)

The matrices \(J_1\) and \(J_2\) are given by

\[
J_1 = \begin{pmatrix}
-\frac{1}{2T} (1 - 2\kappa \cos C_p) & -\frac{2q_x}{1 + q_x T} \\
-2\kappa \cos C_p & -2\kappa \sin C_p & 2\alpha q_x \\
2\kappa \sin C_p & -2\kappa \cos C_p & -2q_x
\end{pmatrix},
\]

(4.4)

\[
J_2 = \begin{pmatrix}
-\frac{1}{2T} (1 - 2\kappa \cos C_p) & -\frac{2q_x}{1 + q_x T} \\
-2\kappa \cos C_p & -2\kappa \sin C_p & 2\alpha q_x \\
2\kappa \sin C_p & -2\kappa \cos C_p & -2q_x
\end{pmatrix}.
\]

(4.5)

The Jacobian \(J\) therefore decomposes the phase space into two subspaces spanned by the coordinates \((q_x, N_S)\) and \((q_y, q_z, N_D)\) respectively. At the pitchfork Hopf bifurcation it is required that \(J\) has one eigenvalue at zero and an additional pair of complex conjugate eigenvalues on the imaginary axis. However, we observe that for \(\kappa < 1/2\) both eigenvalues of \(J_1\) have negative real part. Therefore it follows that at the pitchfork Hopf bifurcation all three eigenvalues of \(J_2\) have zero real part. This in particular implies that \(\text{tr}\ J_2 = 0\) which for \(0 < \kappa < \frac{1}{2}\) gives rise to the condition

\[
\kappa \cos C_p = \frac{1}{4} \left(1 - \sqrt{1 + \frac{4P + 2}{T}}\right) \approx -\frac{2P + 1}{4T},
\]

(4.6)

where the last approximation is correct in first order of \(1/T\). We can then use the condition \(\det J_2 = 0\) to obtain an expression for the pitchfork line in Fig. 27 as

\[
\kappa(2P + 1) = (\alpha \sin C_p - \cos C_p)(1 - 2\kappa \cos C_p)(P + \kappa \cos C_p).
\]

(4.7)

Eliminating \(C_p\) in the above equation using (4.6) yields a quadratic equation in \(\kappa^2\) which gives two solutions for positive \(\alpha\). For the first non-vanishing order in \(1/T\) we find the following expressions for the location of the points PH1 and PH2 in the \((C_p, \kappa)\) parameter plane,

\[
(C_p, \kappa)_{PH1} \approx -\cos^{-1} \left[\frac{\alpha P}{\sqrt{1 + \alpha^2} \left(\frac{\alpha P}{4T} + \frac{2P + 1}{4T}\right)}\right]
\]

\[
(C_p, \kappa)_{PH2} \approx -\cos^{-1} \left[\frac{\alpha}{\sqrt{1 + \alpha^2}} \left(\frac{\sqrt{1 + \alpha^2} (2P + 1)}{4T} - \alpha\right)\right]
\]
Above expressions are valid for the in-phase one-colour states. The co-dimension two points PH1 and PH2 seen in Fig. 27 correspond to the in-phase PH1 and the anti-phase PH2. To obtain the corresponding expressions for anti-phase states, the phases $C_p$ can be shifted by $\pi$ in accordance to parameter symmetry (4.1).

4.B small non-zero delay

It was desirable to include delay in the dynamical states to more realistically model two interacting lasers. Fig 15 and Fig. 19 and for all optical switch designs in the succeeding Chap. 5 all have small non-zero delay. A problem arose whereby it was sometimes guesswork when choosing exact parameters for small regions of Fig. 27. The following was used to estimate the permitted parameter values for different delay and motivated the choice of $\tau = 0.2$ to allow region 7 of Fig. 27 to exist in our high coupling study.

Laser rate equations for single laser mutually coupled with a certain amount of the lasers’ light entering the others’ cavity.

$$\dot{E}_1(t) = (1 + i\alpha)N_1(t)E_1(t) + \kappa e^{-iC_p}\tau E_2(t - \tau) - i\Delta E_1(t) \tag{4.8}$$

$$\dot{E}_2(t) = (1 + i\alpha)N_2(t)E_2(t) + \kappa e^{-iC_p}\tau E_1(t - \tau) + i\Delta E_2(t) \tag{4.9}$$

$$T\dot{N}_1(t) = P_1 - N_1(t) - (2N_1(t) + 1)|E_1(t)|^2 \tag{4.10}$$

$$T\dot{N}_2(t) = P_2 - N_2(t) - (2N_2(t) + 1)|E_2(t)|^2 \tag{4.11}$$

variables $E, N$ functions of $t$, $\{P, T\} \in \mathbb{R}$, $\{\alpha, K\} \in \mathbb{C}$. $K = \kappa e^{-iC_p}$ is the complex injection, $\alpha = 1 + i\alpha'$ where $\alpha'$ is the line enhancement factor.

Assuming sufficiently small positive $\tau$ and using the T.E to the first order $E(t - \tau) = E(t) - \tau \dot{E}(t) + O(\tau^2)$, we write

$$\dot{E}_1(t) = [\alpha N_1(t) - i\Delta] E_1(t) + K E_2(t) - K\tau \dot{E}_2(t) + O(\tau^2)$$

$$\dot{E}_2(t) = [\alpha N_2(t) - i\Delta] E_1(t) + K E_2(t) - K\tau [\alpha N_2(t) + i\Delta] E_2(t) - K^2 \tau E_1(t - \tau) + O(\tau^2)$$

Using the T.E a second time gives the system of autonomous differential equations which approximates the delayed system (4.8)-(4.9) for small $\tau$.

$$\dot{E}_1(t) = [\alpha N_1(t) - i\Delta - K^2 \tau] E_1(t) + K[1 - \tau \alpha N_2(t) - i\tau \Delta] E_2(t) + O(\tau^2) \tag{4.12}$$

$$\dot{E}_2(t) = [\alpha N_2(t) + i\Delta - K^2 \tau] E_2(t) + K[1 - \tau \alpha N_1(t) + i\tau \Delta] E_1(t) + O(\tau^2) \tag{4.13}$$

The above approximations are valid only for small enough $\tau$. System is completed by (4.10)-(4.11). We omit $O(\tau^2)$ and the $t$ dependence in what follows as no ambiguity exists that all variables are functions of $t$.

We make the following coordinate transformation where CW solutions of system (4.8)-(4.11) are equilibria. Importantly this the new coordinate set contains no singularities.

$$q_x + iq_y = 2\dot{E}_1 E_2 \quad q_z = |E_1|^2 - |E_2|^2 \tag{4.14}$$

All variables are dependant on $t$. Using the coordinate transformation above, the system becomes

$$q_x = q_x (N_1 + N_2) + \alpha' q_y (N_1 - N_2) - 2\Delta q_y + 2\kappa R \cos C_p + \tau \kappa (2q_z \Delta \sin C_p - 2\kappa \pi \cos 2C_p - [R(N_1 + N_2) + q_z (N_1 - N_2)] (\sin C_p + \alpha' \cos C_p)) \tag{4.15}$$
\[
q_y = q_y (N_1 + N_2) - \alpha' q_z (N_1 - N_2) + 2\Delta q_z - 2\kappa q_z \sin C_p + \tau \kappa \left\{ 2R \Delta \cos C_p - 2\kappa q_y \cos 2C_p + [q_z (N_1 + N_2) + R (N_1 - N_2)] (\sin C_p - \alpha' \cos C_p) \right\} \\
q_z = q_z (N_1 + N_2) + R (N_1 - N_2) + 2\kappa q_y \sin C_p + \tau \kappa \left\{ -q_z \sin 2C_p + q_x (N_1 - N_2) (\cos C_p + \alpha' \sin C_p) - q_y (N_1 + N_2) (\sin C_p - \alpha' \cos C_p) \right\}
\]

This reduces 4D to 3. It’s noted that stability detection by AUTO [22] can no longer be relied upon and all line are plotted as solid lines in Fig. 38.

Figure 37: Same as Fig. 27 but with bifurcation lines that act on unstable states also included. The injection in Eq. (2.3) \( \kappa e^{iC_p} \) is considered a complex injection where \( K_1 = \cos(C_p) \) and \( K_2 = \sin(C_p) \). Codimension 1 bifurcations that act on unstable dynamical states are included as solid lines.
Figure 38: Bifurcation diagrams created by varying two parameters, with the coupling phase as the abscissa and the coupling strength as the ordinate. The bifurcation lines and points where obtained using continuation software AUTO. The top shows bifurcation lines at zero delay between the lasers with the dot-dashed lines acting on unstable states. The bottom diagram approximates the picture at \( \tau = 0.1 \), no stability information was possible so all lines are drawn solid.
5 All-Optical Switching

The realisation of stable, fast and scaleable all optical memory elements would significantly increase
the scope of photonic devices and has therefore attracted considerable interest in the laser com-

munity. Examples of optical memory designs include hetero-structure photonic crystal lasers [19],
coupled micro-ring lasers [30] and dual mode semiconductor lasers with delayed feedback [15]. In
this chapter, we propose a conceptually simple all optical memory element on the basis of two

closely coupled single-mode lasers.

5.1 Optical Memory Unit 1

Memory units require at least two stable states and a mechanism to switch between them. In the

previous sections, a number of different parameter regions displaying multi-stabilities were dis-
covered for closely coupled lasers. In particular, in section 3.2 corresponding to region 4 of Fig. 27
we established the presence of symmetry-broken two-colour states. Switching between them is
achieved via optical injection from two master lasers as shown schematically in Fig. 39(a). Initially
the lasers are in a symmetry-broken state with an optical spectrum displayed in the top left panel
of Fig. 40. The oscillations in the magnitude of the electric fields, shown in the graph beneath,
are due to the beatings between the two optical frequencies (Sec. 3.2). Between 5 ns and 6 ns an
optical pulse from a master laser is injected into laser 1 which causes laser 1 to lock to the external
frequency and for its carrier density to significantly reduce. This is shown in the top panel of
Fig 41, where we denote the point “A” at approximately 50 ps when the carrier density of laser
1 is pushed beneath that of laser 2. After the pulse is switched off at 6 ns, a transient of about
3 ns is visible in the bottom panel of Fig. 40 for the coupled lasers to completely settle to the twin
symmetry-broken two-colour state which corresponds to the two lasers having been exchanged
(top centre panel of Fig. 40). The contrast ratio between the two stable states is relatively large.
For example, the intensity of the colour with higher frequency changes by a factor of more than four.
One important aspect from an application point of view is the ability to switch between two states with a very short external optical pulse and we therefore define the write time as the minimum pulse duration to ensure switching. In order to demonstrate that a shorter pulse is sufficient to trigger the switch, we inject at 13 ns a pulse of 50 ps duration into laser 2. In the lower panel Fig 41, we see that $N_2$ is indeed pushed below $N_1$ during the pulse. After the pulse is switched off, both $N_1$ and $N_2$ oscillate strongly but $N_2$ remains consistently below $N_1$. Therefore the write time is determined by the minimum time needed to reduce the carrier density of one laser below that of the other laser which for our setup is approximately 50 ps. We have thus demonstrated a basic mechanism for an all-optical memory element with a large contrast ratio, a short write time and a low coupling strength between the lasers.

5.2 Optical Memory Unit 2

For certain applications it may be desirable to inject into only one of the two lasers as in Fig. 39(b). To achieve this, we choose parameters consistent with region 5 of Fig. 27. In this region, in addition to the two-colour symmetry-broken states of the previous paragraph, a one-colour symmetric state is also stable. In Fig. 42, these states form the basis for the optical memory unit. Initially the two lasers start in a degenerate state with the same amplitude and the same single frequency in both lasers. Using a pulse with a positive detuning relative to the central frequency (+83 GHz in the case of Fig. 42) the symmetric state of the two lasers symmetry breaks to a two-colour state. As the number, position and intensities of frequencies change, the two states can be easily distinguished which is desirable from an application point of view.

To discuss how quickly the two states can be ascertained, we introduce the read time which is the minimum duration needed to differentiate optically between the two states after the injection has turned off. In Fig. 41 for the first optical switch after the external injection was removed at 6 ns and 13.05 ns, large amplitude oscillations in the carrier densities at a frequency consistent with the relaxation oscillations for the coupled system (5.7 GHz) ensue. These oscillations are directly related to the length of the transient observed in the optical fields for the system to completely settle to the twin symmetry-broken two-colour state. We stress that one does not need to wait for all relaxation oscillations in the system to die out before reading. Small amplitude high frequency oscillations (40 GHz) which are due to the beating between the two optical colours (see Sec. 3.2) are observable before each switching event and are discernible in Fig. 41 as small deformities in the larger amplitude oscillations within 1 ns of the external injection being turned off. In Fig. 43, two Fourier modes consistent with the peak frequencies of $-32$ GHz and $-83$ GHz are traced out for each switching event for the second optical switch. All other frequencies are filtered out. During the first injection episode from 5 ns to 6 ns, the frequency centred at $-83$ GHz is turned off within 400 ps. We also observe that after injection, the frequency at $-32$ GHz has reached a stable intensity within 100 ps. To switch back to the symmetric one-colour state, a pulse with negative detuning ($-38$ GHz in the case of Fig. 42) is injected into laser 1. In the lower panel of Fig. 43 the frequency component at $-83$ GHz relaxes to its equilibrium value within 1.5 ns. The frequency at $-34$ GHz is completely off within 100 ps. This constitutes a robust memory element where we inject into one laser only. An external injection pulse with positive detuning causes the coupled lasers to enter a symmetry-broken state whilst negative detuning causes the coupled lasers to enter a symmetric state. Compared to the switching scenario in the previous paragraph (Fig. 40) the two lasers are more strongly coupled but the contrast ratio between the two states is greater.
5.3 Optical Memory Unit 3

Two semi-conductor laser diodes, mutually coupled via light of each entering the cavity of the other after a small delay, were modelled by a system of non-linear rate equations. Using a new reduced model, a bifurcation diagram, Fig. 27 was obtained by varying the coupling strength and coupling phase. This separated parameter space into regions of distinct dynamics. Symmetry-broken one-colour states were shown to be stable, where each laser has significantly different intensity but are locked to the same single frequency (cf. Fig. 44). Symmetric and symmetry-broken two-colour states were also found and explained as slow-fast dynamical behaviour. Due to the ability to exchange both lasers, symmetry-broken states always exist in pairs. Overlap with regions of symmetric one-colour states creates regions of tri-stability.

Figure 44 shows a third example of all-optical memory element. As its basis, it uses a bi-stability between symmetric one-colour and symmetry-broken one-colour states which is consistent with parameter from region 7 of Fig. 27. Switching is achieved via optical injection from a master laser injecting into one of the coupled lasers as in Fig. 39(b). Initially the two lasers start in a degenerate state with the same intensity. Using a pulse of a positive detuning relative to the free running frequency, the symmetric state symmetry breaks. As the frequency and intensity changes, the on and off states can be easily distinguished, which is desirable from an application point of view. To switch back a pulse of negative detuning is injected into one of two lasers. This provides a robust switching where negative detuned frequency causes the coupled lasers to become or remain symmetric and positive detuned frequency causes the lasers to enter or stay in a symmetry-broken one-colour state.
Figure 40: Optical switching between symmetry-broken two-colour states for a two laser configuration as outlined in Fig. 39(a) for parameters $\tau = 0.2$, $\kappa = 0.1$, $C_p = 0.35\pi$. The large bottom panel shows the magnitude of the electric field in both lasers. Laser 1 in red and laser 2 mirrored underneath in blue. The central black line dividing the two shows the injection strength and duration of the master lasers. At 5 ns a pulse of 1 ns is injected from a master laser into laser 1. At 13 ns a pulse of 50 ps from a master laser is injected into laser 2. The top three panels show the corresponding frequency states. The black arrow indicates the frequency of injected light which is 19.5 GHz larger than the lasers' free running frequency.
Figure 41: Diagrams showing the dynamics of the carrier densities for both lasers during the first switching event (top panel) and second switching event (bottom panel) of Fig. 40. Black vertical lines mark the time when the external injection is turned on and then off. Point labelled “A” indicates where the carrier density of laser 1 becomes less than laser 2.
Figure 42: Optical switching between a symmetric one-colour state and a symmetry-broken two-colour state as sketched in Fig. 39(b) for parameters $\tau = 0.2$, $\kappa = 0.2$, $C_p = 0.25\pi$. Panel layout as in Fig. 40. At 5 ns a pulse of 1 ns with frequency offset of +83 GHz is injected from the master laser into laser 1. At 20 ns a pulse of 1 ns with frequency offset of -32 GHz is injected from the master laser into laser 1.
Figure 43: Intensity plot tracing frequencies $-32$ GHz and $-83$ GHz for the un.injected laser 2 during the first switching event (top panel) and second switching event (bottom panel) of Fig. 42. A Fourier transform using a Hann window is executed every 100 μs.
Figure 44: Optical switching between symmetry-broken one-colour states and symmetric one colour states for a two laser configuration as outlined in Fig. 30(b) for parameters $\tau = 0.2$, $\kappa = 0.3$, $C_p = 0.33\pi$. The large left panel shows the magnitude of the electric field in both lasers. Laser 1 in red and laser 2 plotted after in blue. The black line starting at zero intensity shows the injection strength and duration of the master lasers. At 5 ns a pulse of 1 ns is injected from a master laser into laser 1. At 20 ns a pulse of 1 ns from a master laser is injected for a second time into laser 1. The right most panels show the corresponding frequency states. The black arrow indicates the frequency of injected light which is $+121$ GHz in the first switching event and $-59$ GHz in the second switching event from the lasers’ free running frequency.
6 Conclusions

A comprehensive study of the dynamics of two interacting semi-conductors lasers is presented. A detailed knowledge of the bifurcations and their boundaries of closely coupled lasers may open several technological applications. We propose three all-optical switch candidates.

6.1 High Coupling

For strong coupling, a comprehensive overview of the bifurcation scenarios in a system of two closely coupled single-node lasers is provided. For moderate to high coupling strength the four characteristic stable states are symmetric one-colour, symmetry-broken one-colour, symmetric two-colour and symmetry-broken two-colour states. In section 2.3 we introduce a new coordinate representation which accounts for the $S^1$ symmetry in the system without creating unnecessary singularities. This allows us to study the bifurcation structure of this system using conventional numerical continuation techniques.

Our results show that the bifurcations between the various stable states are organised by a number of codimension two bifurcation points which are identified with reference to the literature. In particular it is found that the interplay between a pitchfork Hopf, a saddle-node Hopf and a saddle-node pitchfork codimension two points give rise to regions of multi-stabilities.

6.2 Low Coupling

The study of two weakly coupled lasers is introduced by considering two free-running lasers. At initial coupling the lasers lock to either an in-phase or anti-phase symmetric one-colour states. Symmetry-broken one-colour states emerge from considering one laser initially “on” and the other laser initially “off”. The weakly coupled limit is defined in which a description of the dynamics at instantaneous coupling suffices. While discussing a loss of stability of anti-phase one-colour states at a Hopf bifurcation, it is shown that the plane in which the limit cycle is born heavily influences the electromagnetic spectra of the laser. This Hopf bifurcation which leads to mode-beating at high coupling in Sec. 3.2 leads to undamped relaxation oscillations in Sec. 3.3 at low coupling.

A full bifurcation picture, Fig. 33 is developed for identical lasers at low coupling valid for moderate delay-times. Codimension one bifurcation lines split parameter space into regions of distinct dynamical behaviour. In particular at low coupling we identify regions of stable in-phase and anti-phase one-colour states, symmetry and symmetry-broken undamped relaxation oscillations, symmetric and symmetry-broken quasi-periodic behaviour, and symmetric and symmetry-broken chaotic dynamics. Several regions of bi-stability exist among the states. The codimension two bifurcation points which anchor the codimension one bifurcation lines are identified with reference to the literature.
6.3 Memory Element

In chapter 5 we have shown that two closely coupled identical lasers can operate as an optical memory element and have provided three specific examples. The high degree of multi-stability discovered in the previous chapters 3 and 4 enables many other designs. As the multi-stabilities persist in the limit of $\tau \to 0$, the distance between the lasers can be reduced as far as is technologically possible. Memory units of this kind are therefore open to miniaturisation and allow integrability. For the optical switch of Fig. 40, a fast write time of 50 ps was demonstrated in the second switching event and was discussed with reference to the carrier density in Fig. 41. A significant speed increase on this number may be possible by increasing the injection strength of the external pulse or a fine tuning of the coupled lasers’ parameters. For the optical switch of Fig. 42, we showed via Fig. 43 that it is possible to distinguish optically between the states within a read time of 100 ps after the external injection was turned off. Again significant improvements on this number may be possible. Indeed the larger the frequency separation between states the shorter the time needed to differentiate between them. Therefore choosing parameters consistent with larger frequency separations which normally occur at higher coupling strength may substantially decrease the read time. We conclude that closely coupled lasers offer a promising approach for the realisation of scaleable and fast all optical memory elements.
7 Outlook

The research work of this thesis is not complete. We failed to obtain a full dynamical systems' picture for two mutually coupled single mode lasers even in the instantaneous limit, $\tau = 0$. This happened because the equations that govern two interacting lasers are complicated and the project simply ran out of time. To complete the zero delay milestone, a detailed knowledge of the dynamics for the two identical lasers under medium coupling needs to be added. This would essentially connect the two quasi-independent dynamical system studies of low and high coupling strengths of the thesis. For low coupling we saw that properties of the free-running laser such as the relaxation oscillation frequency dominates the dynamics. For high coupling the simple network topology of the two identical mutually connected nodes is more important whilst specifics of the node, such as the carrier density, play only a minor role. In the medium coupling range it is an expected there to be a complicated interplay between the effects of the network and that of the node. Therefore it may be speculated that chaotic and complex dynamics which is observed in timetraces shown in Fig. 45 may be present. It is intended that this chapter both records what we’ve learnt about the medium coupling dynamics and steers the reader towards some interesting open questions as of 2014. It will be seen that while several key facts may need to be checked, a considerable initial investigation has already been performed and therefore it might be hoped that a full picture may in fact be forthcoming with modest time investment. Perhaps counterintuitively, as complex dynamics is normally avoided when devising applications, the dynamics observed for moderate coupling strengths may in fact be application rich. A highly competitive all-optical true random number generator based upon these dynamics is proposed in the final section.

7.1 Medium Coupling

Fig. 46 displays the fullest but incomplete understanding we have that connects the high and low dynamical system studies of this thesis. In the diagram, the coupling strength on the ordinate is log-scaled. We leave out an explanation of the bifurcation lines but the nomenclature and colour codes are the same as other figures of the thesis. Direct comparison between the lower panel and the low coupling bifurcation diagram Fig. 33 should be clear. The initial coupling region, although visible, cannot extend to zero coupling $\kappa = 0$ due to the logarithm and is cut at $\kappa = 0.0005$. The top panel is easily indentified as the bifurcation picture at high coupling strengths, Fig. 27. The centre panel which extends approximately an order of magnitude of the coupling strength parameter $\kappa$ and termed medium coupling was not addressed in this thesis. Several periodic orbits from the low coupling study where followed with AUTO [22] into the medium coupling range showing ever increasing periodicity, as shown in Fig. 47\footnote{Visibly those resemble the picture for Shil'nikov bifurcations where the period of a cycle approaches a saddle-focus homoclinic bifurcation with positive saddle quantity as in Ref. [64]. This may not be a textbook example as the limit cycles lose and gain stability with alternating fold and period doubling or torus bifurcations.}. Limit cycles of very high but constant period were then followed with the two parameters and displayed by the thick olive-brown line. For most points along the curve a homoclinic bifurcation was detected which is represented by the thin dark green line. Whether these are stable structures or not was not ascertained. The black line in Fig. 46, as in Fig. 33, bounds an area of positive maximal Lyapunov exponent. Perhaps naïvely
one could get away with labelling this region as chaotic and move on. But in the opinion of this author this would leave many questions unsatisfactorily unanswered. An improved understanding of the mechanism that creates the chaotic dynamics of the medium coupling region may reveal itself if the full Lyapunov spectrum with vector directions were obtained.

Several routes to chaos were not fully verified. In Fig. 46 from the region of symmetry-broken undamped relaxation oscillations, Sec. 3.3 (low coupling centre region roughly bounded by $0.2\pi < C_p < 0.6\pi$), after increasing the coupling strength at least five period doubling bifurcations were detected with AUTO before entering the chaotic region. This was also observed using a min-max diagram. However the expected period doubling cascade was not observed and the Feigenbaum scaling appeared not to be obeyed. Therefore, we expect this not to be the well known period doubling route to chaos. If confirmed this would differ from the single laser with optical injection scenario [102, 52]. To the lower right of diagram Fig. 46 where symmetric quasiperiodic states are stable (cf. Fig. 21 of Sec. 3.4), a form of torus breakdown seems to occur. A timetrace of a typical state near the boundary region is shown in Fig. 48. As can be seen in the Fourier spectra of this diagram the central peak with sidebands spaced at the relaxation oscillations of the free running lasers are still clearly present. A host of new frequencies has turned on, indicative of chaos. Similarly for symmetry-broken quasiperiodic states (lower left of Fig. 46; cf. Fig. 22 of Sec. 3.4) may enter the chaotic region by a form of torus breakdown. The timetrace in Fig. 49 is similar to Fig. 48 but the long term behaviour in the peaks in the Fourier spectra have different amplitudes at the same frequency between the lasers. Therefore we suspect that the dynamics observed in Fig. 48 and Fig. 49 may be due to a symmetric and symmetry-broken chaotic attractors. A clear boundary between these two chaotic states at $C_p = 0.6\pi$ extends into the chaotic region. If confirmed, the exact mechanism that leads to this symmetry-breaking would be of broad interest. For higher coupling strength the chaotic spectra loses its association with the undamped relaxation oscillations (cf. Fig. 45). Decreasing the coupling strength for a symmetry-broken two-colour state (cf. Fig. 16 of Sec. 3.2), at least two period-doubling bifurcations occur before entering the chaotic region of Fig. 46. No attempt to verify a period-doubling cascade route to chaos was made.

Outside the chaotic region of the medium coupling band in Fig. 46, termination of the red PLC and purple SNLC lines needs to be properly ascertained. Investigation may yield a sizeable region of multi-stability between limit cycles and fixed point states prompting the possibility that further applications may be devised.

### 7.2 Random Number Generator

Random numbers play an important behind-the-scenes role in our modern lives. During this last year issues of cryptography and private communication were highlighted in the public media. In scientific programming the ability to generate a large number of random numbers quickly has for many decades been vital across many disciplines including: weather and climate modelling, epigenetics, fluid mechanics, quantum chromodynamics, computational biology, astrophysics, economics, financial market prediction and much more. For all form factors and all operating systems, from our handheld smartphones to the today's top supercomputers, a random number generator is a key component. Often these are algorithm-based and provide pseudo-random numbers which are deterministic. This may be desirable for some applications. A true random number generator TRNG is based upon a physically occurring source of entropy [41] and is therefore completely unpredictable. Clearly this is a financial imperative for online casinos and an added security in cryptography. Several TRNGs are commercial available. For example, the TectroLabs TL200 TRNG released mid 2014 retails at US$349 and provides random bits at 2.0 MB/s. Chaotic semiconductor lasers
were proposed and lab demonstrated as possible entropy source for a random number generator
RNG capable of Gb/s generation rate [114]. The element of randomness is provided from quantum
fluctuations in the spontaneous emission of photons which are amplified by chaotic dynamics to a
macroscopic fluctuating signal [78]. Several designs exist but most use a standard semiconductor
laser with feedback from a mirror to provide the chaotic source [93, 55, 40]. A finite distance
between laser and reflector is required to induce the chaotic response. For example [40] has a
photon roundtrip of 12 ns which equates to a distance of 1.8 m between laser and mirror. Instead
we propose using two closely coupled semiconductor lasers where a second laser takes the place
of the mirror. Although a speed doubling may naturally arise from the use of a second laser, the
principal advancement is that the chaotic dynamics needed for the random bit generation persist
as distance between the two lasers is reduced to zero. Therefore this scenario allows for much
greater miniaturisation whilst retaining the ultrafast speeds of the former.

In this paragraph we justify using closely coupled lasers as a TRNG using the theoretical model.
It is cautioned that only a physical realisation of this two laser setup in a laboratory environment
would measure the true nature of randomness. At best using the rate equation model (2.3), cer-
tain statistical properties of the chaotic attractor are obtainable. Nevertheless we aim to provide
a simple viability demonstration so that further experimental investigations are encouraged. First,
in order to most readily link to current computer technology, for the purpose here it was decided
that it is preferable to generate random bits. Random numbers in decimal or other numeral sys-
tems may allow considerably faster speeds. For binary, a convenient way to define what is 1 and
what is 0 is needed. Many approaches are possible such as using the derivative of the output [93]
but for this simple demonstration, 1 is defined when laser 1 has more optical output than laser 2,
otherwise the system is defined as 0. Fig. 50 makes this more clear by showing the distribution
of outputs of a laser and its respective binary definition. For there to be an equal probability
of 1s and 0s, Fig. 46 is consulted and parameters consistent with a symmetric chaotic attractor
are chosen. Relatively weak coupling between the two lasers is required to attain chaos, leaving
a strong random signal which may be desirable from an application point-of-view. As we wish to
demonstrate a 1 Gb/s stream of random bits, the output of each laser is sampled every 1 ns. The
LHS of Fig. 51 shows a simple graphical test made popular by [41]. The absence of patterns passes
a basic test of randomness [4]. Two widely accepted test suites for random numbers [73] and [100]
are publicly available. The RHS of Fig. 51 contains a table of ten selected statistical tests for the
random bits produced by this method. The values obtained are consistent with what is expected
from a sequence of random numbers.

We stress that many design options and flavours are possible with the ultimate one depending on
what technologies are to be incorporated. For example, to make an all-optical TRNG [68, 94] it
may be desirable to implement the RNG in frequency space [5] and/or in a higher numeral system.
Alternatively, for a RNG that connects with current electronics the fluctuating optical signal can
be easily converted using a photodetector and analog-to-digital-converter, as in [93, 40, 129]. Ref.
[8] boasts an amazing 140 Gb/s TRNG based upon a single laser with mirror manufactured on a
photonic integrated circuit PIC. The DFB laser used has a length of 0.3 mm but an external cavity
in excess of 1 cm. The two closely coupled lasers’ approach proposed here would eliminate almost
all of the required size. Additionally, the realisation of two lasers on a PIC is expected to scale
linearly allowing for even faster speeds. It is interesting to speculate that these sorts of speeds
would lead to a paradigm shift in RNG market. Currently an independent TRNG component is a
niche product desired only by applications in which unpredictability is crucial. A larger customer
base which includes the vast majority of theoreticians, scientists and programmers who use random
numbers in models which incorporate noise effects, use Monte Carlo methods, or stochastics favour
pseudo-RNG because with current technology they are faster. There is no reason why this should be so. The promise of an ultrafast TRNG may attract a sizeable new audience looking to speed up their calculations by removing a considerable load from their CPUs by delegating random number generation to a separate external or onboard component.
Figure 45: Timetraces showing the dynamics for $\tau = 0, \kappa = 0.02, C_p = 0.5\pi$. 
Figure 46: Parameter map for two closely coupled lasers encompassing both the high coupling, Sec 4.1 and low coupling studies, Sec. 4.2. Coupling strength on the vertical axis is logscaled. New lines not appearing in Figs. 27 and 33 are: olive-brown lines are very high period limit cycle (see text), dark-green is a homoclinic bifurcation (see text), and black lines bounds areas of positive Lyapunov exponent.
Figure 47: The period of the limit cycles for symmetric undamped relaxation oscillations (red) and for symmetry-broken undamped relaxation oscillations (blue) is plotted as the coupling strength $\kappa$ is varied with constant coupling phase $C_p = 0.3\pi$. Solid lines show stable limit cycles. Only bifurcations (black dots) that affect stable dynamics are labelled.
Figure 48: Timetraces showing the dynamics for $\tau = 0$, $\kappa = 0.004$, $C_p = 0.75\pi$. Parameters and labelling consistent with a point in Fig. 33.
Figure 49: Timetraces showing the dynamics for $\tau = 0$, $\kappa = 0.004$, $C_p = 0.4\pi$. Parameters and labelling consistent with a point in Fig. 33.
Figure 50: A histogram of the difference in magnitudes of the electric field of the lasers for parameters \( \tau = 0, \kappa = 0.02, C_p = 0.5\pi \) (cf. Fig. 45; region of Fig. 46 where chaotic attractor is expected to be invariant with respect to the \( Z_2 \) symmetry).

![Histogram](image)

<table>
<thead>
<tr>
<th>Statistical test</th>
<th>P-value</th>
<th>Proportion</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>0.350485</td>
<td>10/10</td>
<td>Success</td>
</tr>
<tr>
<td>Block Frequency [128]</td>
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<td>10/10</td>
<td>Success</td>
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<tr>
<td>Cumulative sums</td>
<td>0.739918</td>
<td>10/10</td>
<td>Success</td>
</tr>
<tr>
<td>Runs</td>
<td>0.534146</td>
<td>10/10</td>
<td>Success</td>
</tr>
<tr>
<td>Longest Run</td>
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<td>10/10</td>
<td>Success</td>
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<td>Rank</td>
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<td>10/10</td>
<td>Success</td>
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<td>FFT</td>
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<td>10/10</td>
<td>Success</td>
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<tr>
<td>Overlapping Template [9]</td>
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<td>Success</td>
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<td>Approximate Entropy [10]</td>
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<td>10/10</td>
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<td>Linear Complexity [500]</td>
<td>0.122325</td>
<td>10/10</td>
<td>Success</td>
</tr>
</tbody>
</table>

Figure 51: Randomness tests for bits generated for parameters \( \tau = 0, \kappa = 0.02 \) and \( C_p = 0.5\pi \) (cf. Fig 45 and Fig 50). Left: a static 512 \( \times \) 512 pixel image. Right: a table for 10 statistical test from the NIST with significance level \( \alpha = 0.01 \) using 10 samples of 1 Mb. Numbers in square brackets beside statistical test constitute the default block sizes used.